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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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Again $\frac{\partial S}{\partial \alpha}, \frac{\partial \phi}{\partial \eta}$ contain only terms which vanish with α.

$$\therefore \qquad \frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial S}{\partial \eta} = \frac{\partial \phi}{\partial \eta} \cdot \frac{\partial S}{\partial \alpha} = \frac{1}{2} \frac{\pi}{\Delta} S'_0 + \ldots\ldots$$

Consequently, $|\alpha|$ being small, the condition for stability becomes

$$M'. B\alpha. \frac{1}{2} \frac{\pi}{\Delta} S'_0 > 0;$$

or, since B and S'_0 are both negative (28.1), (29.1), the condition is

$$M' \alpha > 0. \qquad (30.15)$$

Now
$$M' = \frac{dM_\alpha}{d\alpha}$$

and $\alpha \frac{dM_\alpha}{d\alpha}$, when $|\alpha|$ is small, has the same sign as $(M_\alpha - M_0)$. Therefore we see that the necessary and sufficient condition for stability is that

$$M_\alpha > M_0. \qquad (30.16)$$

Thus the necessary and sufficient condition that the pear-shaped figures should be stable is that the angular momentum of the pear-shaped figures should be g eater than that of the Elliptic Cylinder at the point of bifurcation from which it is derived.

We have already shown that if A_3 is positive, $M_\alpha < M_0$; we therefore see that in this case the pear-shaped figures are unstable.

We therefore proceed to evaluate the constant A_3 and show that it is positive.

PART IV.

The Evaluation of the Necessary Constants.

§ 31. In the last part we have shown that there are essentially only two constants that we have to evaluate in order to determine the stability or the instability of the pear-shaped figures of equilibrium, namely, B, A_2 or B, A_3 according as m is even or odd. Further we have already shown from more or less *a priori* considerations that B is negative—being in fact equal to

$$-\frac{1 + t^{m-2}}{\Delta} \equiv -\frac{1 + t^{m-2}}{ab} = -\frac{1}{ab}\left[+ 1 \left(\frac{a-b}{a+b}\right)^{m-2}\right]. \quad (29.1)$$

We also said that, when m is even, A_2 is always positive and that, when m is odd, A_2 is identically equal to zero but that A_3 is positive at

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least in the case $m = 3$ and probably so in general. Let us now proceed to prove these results.*

Consider now the constant B.

We have $\quad B =$ co-efficient of $\alpha\eta$ in A

$$= \text{co-efficient of } \alpha\eta \text{ in } \frac{1}{\pi}\int W \cos m\psi \, d\psi.$$

Noting that

$$\zeta = \sum_{r+s>0} \zeta_{rs}\, \alpha^r \eta^s,$$

$$W = \eta(\rho + q\cos^2\psi + \zeta) + U_2 + U_3 + \ldots\ldots,$$

and that U_n is of order n in ζ, ζ', we see therefore that

$$B = \text{co-efficient of } \alpha\eta \text{ in } \frac{1}{\pi}\int [\eta(\rho + q\cos^2\psi + \zeta) + U_2] \cos m\psi \, d\psi$$

$$= \frac{1}{\pi}\int \zeta_{10} \cos m\psi \, d\psi + \text{co-efficient of } \alpha\eta \text{ in } \frac{1}{\pi}\int U_2 \cos m\psi \, d\psi.$$

But by (7.3) or (23.1)

$$U_2 = \lim_{\substack{v = \rho \\ u \to \rho - 0}} \left[\frac{1}{2}\frac{\partial}{\partial v}\int K\zeta'^2 \, d\psi' + \zeta \frac{\partial}{\partial u}\int K\zeta' \, d\psi'\right]$$

wherein $\quad K \equiv K(uv) = -\log R^2(uv) \dfrac{H(v\psi')}{2\Delta(v)}$. $\quad$ (6.3).

In this we have to substitute

$$\zeta = \zeta_{10}\,\alpha + \zeta_{01}\,\eta + \zeta_{20}\,\alpha^2 + \zeta_{11}\,\alpha\eta + \zeta_{02}\,\eta^2 + \ldots\ldots$$

$$\zeta' = \zeta'_{10}\,\alpha + \zeta'_{01}\,\eta + \zeta'_{20}\,\alpha^2 + \ldots\ldots$$

where, as before, the dashes indicate that the argument is ψ' and not ψ.

The co-efficient of $\alpha\eta$ in U_2 is thus equal to

$$\operatorname{Lim}\left[\frac{1}{2}\frac{\partial}{\partial v}\int K.2\zeta'_{10}\zeta'_{01}\, d\psi' + \zeta_{10}\frac{\partial}{\partial u}\int K\zeta'_{01}\, d\psi' + \zeta_{01}\frac{\partial}{\partial u}\int K\zeta'_{10}\, d\psi'\right].$$

* In evaluating these constants we shall have to use many things that require justification; most of these will be considered in § 39 or in Part V which specially deals with the convergence of the series used.

If in this we use the fact that the ζ_{os} are all constants given by (20.6),

i.e.
$$\zeta_{os} = \frac{1}{S!}\,\frac{d^s\rho}{d\Omega},$$

we get the co-efficient of $\alpha\eta$ in U_2 equal to

$$\zeta_{01} \lim_{\substack{v=\rho \\ u\to\rho-0}} \left[\frac{\partial}{\partial v}\int K\zeta'_{10}\,d\psi' + \zeta_{10}\frac{\partial}{\partial u}\int K d\psi' + \frac{\partial}{\partial u}\int K\zeta'_{10}\,d\psi'\right].$$

Hence we get for (π B) the following expression :—

$$\int \zeta_{10}\cos m\psi\, d\psi + \zeta_{01} \,\mathrm{Lim} \int \left[\left(\frac{\partial}{\partial v} + \frac{\partial}{\partial u}\right)\int K\zeta'_{10}\,d\psi' + \zeta_{10}\frac{\partial}{\partial u}\int K d\psi'\right]\cos m\psi\, d\psi. \quad (31.1)$$

In this we have to use (21.5), (6.3),

i.e.
$$K(uv) = -\log R^2\,(uv)\,\frac{H(v\psi')}{2\,\Delta(v)};\quad H(\rho\psi)\,\zeta_{10} = \cos m\psi.$$

Consider the term:— $\zeta_{01}\int\zeta_{10}\cos m\psi\, d\psi\,.\,\frac{\partial}{\partial u}\int K d\psi'$.

We have

$$\frac{\partial}{\partial u}\,K d\psi' = \frac{\partial^2}{\partial u\,\partial v}\int\int K dv d\psi'$$

$$= \frac{\partial^2}{\partial u\,\partial v}\,[\text{potential of cylinder } v \text{ at point } u]$$

$$= \frac{\partial^2}{\partial u\,\partial v}\left[\text{constant} - \frac{2\pi\,(b_v x^2 + a_v y^2)}{a_v + b_v}\right],$$

where a_v, b_v are the semi-axes of the cylinder v and

$$x = a_u\cos\psi = \sqrt{u+q}\,\cos\psi;\quad y = b_u\sin\psi = \sqrt{u}\,\sin\psi$$

are the co-ordinates of the point at which the potential is taken.

$$\therefore\quad \frac{\partial}{\partial u}\int K d\psi' = -2\pi\,\frac{\partial}{\partial v}\left[\frac{\sqrt{v}\cos^2\psi + \sqrt{v+q}\,\sin^2\psi}{\sqrt{v+q}+\sqrt{v}}\right]$$

$$\therefore\quad \lim_{\substack{v=\rho \\ u\to\rho-0}} \frac{\partial}{\partial u}\int K d\psi' = -2\pi\,\frac{\partial}{\partial\rho}\left[\frac{\sqrt{\rho}\cos^2\psi + \sqrt{\rho+q}\,\sin^2\psi}{\sqrt{\rho+q}+\sqrt{\rho}}\right]$$

$$= -2\pi\,\frac{\partial}{\partial\rho}\left[\frac{a\sin^2\psi + b\cos 2\psi}{a+b}\right].$$

But
$$\frac{\partial X(a,b)}{\partial \rho} = \frac{1}{2a}\frac{\partial X}{\partial a} + \frac{1}{2b}\frac{\partial X}{\partial b}.$$

Therefore the right-hand side

$$= -\frac{2\pi}{2a}\left[\frac{\sin^2\psi}{a+b} - \frac{b\cos^2\psi + a\sin^2\psi}{(a+b)^2}\right] - \frac{2\pi}{2b}\left[\ldots\ldots\right]$$

$$= -\frac{\pi}{ab}\left(\frac{a-b}{a+b}\right)\cos 2\psi.$$

Therefore the term in question is equal to

$$\zeta_{01}\int \frac{\cos^2 m\psi}{H(\rho\psi)}\,d\psi \,. -\frac{\pi}{ab}\left(\frac{a-b}{a+b}\right)\cos 2\psi$$

$$= -\frac{\pi\zeta_{01}}{ab}\left(\frac{a-b}{a+b}\right)\int \frac{\frac{1}{2}\cos 2\psi + \frac{1}{4}\cos 2(m-1)\psi + \frac{1}{4}\cos 2(m+1)\psi}{H(\rho\psi)}\,d\psi\,;$$

or, by (15.5), the term in question is equal to

$$-\frac{\pi\zeta_{01}}{ab}\left(\frac{a-b}{a+b}\right).\frac{2\pi}{ab}\left[\frac{1}{2}\left(\frac{a-b}{a+b}\right) + \frac{1}{4}\left(\frac{a-b}{a+b}\right)^{m-1} + \frac{1}{4}\left(\frac{a-b}{a+b}\right)^{m+1}\right].$$

Also
$$\Omega = \frac{\omega^2}{2} = \frac{2\pi ab}{(a+b)^2}.$$

$$\therefore \quad \frac{d\Omega}{d\rho} = \left(\frac{1}{2a}\frac{\partial}{\partial a} + \frac{1}{2b}\frac{\partial}{\partial b}\right)\Omega = \frac{\pi}{ab}\left(\frac{a-b}{a+b}\right)^2.$$

$$\therefore \quad \zeta_{01} = \frac{1}{1!}\frac{d\rho}{d\Omega} = \frac{ab}{\pi}\left(\frac{a+b}{a-b}\right)^2 \text{ by (20.6).}$$

Therefore the term in question is equal to

$$-\left(\frac{a+b}{a-b}\right).\frac{\pi}{ab}\left[t + \tfrac{1}{2}t^{m-1} + \tfrac{1}{2}t^{m+1}\right]$$

where $t = \dfrac{a-b}{a+b}$ as before.

Consider now the other terms in the expression for πB.

The term $\int \zeta_{10}\cos m\psi\,d\psi = \int \dfrac{\cos^2 m\psi\,d\psi}{H} = \dfrac{\pi}{ab}(1+t^m)$.

The other terms are equal to

$$\zeta_{01} \operatorname{Lim} \left\{ \begin{array}{l} \dfrac{\partial}{\partial v} \displaystyle\int \frac{\cos m\psi'}{\mathrm{H}(\rho\psi')} d\psi' \int \mathrm{K}(uv) \cos m\psi \, d\psi \\ + \dfrac{\partial}{\partial u} \displaystyle\int \frac{\cos m\psi'}{\mathrm{H}(\rho\psi')} d\psi' \int \mathrm{K}(uv) \cos m\psi \, d\psi \end{array} \right\}$$

$$= \zeta_{01} (\mathrm{I}_1 + \mathrm{I}_2), \text{ say.}$$

Consider I_2.

$$\mathrm{I}_2 \equiv \operatorname{Lim} \frac{\partial}{\partial u} \int \frac{\cos m\psi'}{\mathrm{H}(\rho\psi')} d\psi' \int - \frac{\log \mathrm{R}^2(uv)}{2\,\Delta(v)} \mathrm{H}(v\psi') \cos m\psi \, d\psi.$$

Since the differentiation is with respect to u, we may put $v = \rho$ in this and so we get

$$\mathrm{I}_2 = \operatorname*{Lim}_{u \to \rho} \frac{\partial}{\partial u} \int \frac{\cos m\psi'}{2\,\Delta(\rho)} \cdot \frac{2\pi}{m} e^{-m\alpha_\rho} \left(e^{m\alpha_u} + e^{-m\alpha_u} \right) \cos m\psi' \, d\psi'$$

$$= \frac{\pi^2}{m} \mathrm{Q} \frac{d\mathrm{P}}{d\rho} \text{ from (14.2) and (14.3).}$$

Again

$$\mathrm{I}_1 = \operatorname{Lim} \frac{\partial}{\partial v} \int \frac{\cos m\psi'}{\mathrm{H}(\rho\psi')} \frac{\mathrm{H}(v\psi')}{2\,\Delta(v)} d\psi' \int - \log \mathrm{R}^2(uv) \cos m\psi \, d\psi$$

$$= \operatorname{Lim} \frac{\partial}{\partial v} \int \frac{\cos m\psi'}{\mathrm{H}(\rho\psi')} \frac{\mathrm{H}(v\psi')}{2\,\Delta(v)} \cdot \frac{2\pi}{m} e^{-m\alpha_v} \left(e^{m\alpha_u} + e^{-m\alpha_u} \right) \cos m\psi' \, d\psi'$$

$$= \frac{\pi (e^{m\alpha} + e^{-m\alpha})}{m} \int \frac{\cos^2 m\psi'}{\mathrm{H}(\rho\psi')} \frac{\partial}{\partial\rho} \left[\frac{e^{-m\alpha} \mathrm{H}(\rho\psi')}{\Delta(\rho)} \right] d\psi'$$

$$= \frac{\pi \mathrm{PQ}}{m} \int \frac{\cos^2 m\psi}{\mathrm{H}(\rho\psi)} d\psi + \frac{\pi^2}{m} \mathrm{P} \frac{d\mathrm{Q}}{d\rho}.$$

Thus we have the following expression for $\pi\mathrm{B}$:—

$$- \left(\frac{a+b}{a-b} \right) \frac{\pi}{ab} \left[t + \tfrac{1}{2} t^{m-1} + \tfrac{1}{2} t^{m+1} \right] + \frac{\pi}{ab} \left[1 + t^m \right]$$

$$+ \zeta_{01} \left[\frac{\pi^2}{m} \mathrm{Q} \frac{d\mathrm{P}}{d\rho} + \frac{\pi^2}{m} \mathrm{P} \frac{d\mathrm{Q}}{d\rho} + \frac{\pi \mathrm{PQ}}{m} \int \frac{\cos^2 m\psi}{\mathrm{H}(\rho\psi)} d\psi \right].$$

Noting that $t = \dfrac{a-b}{a+b}$ (15.1) and that by (15.5)

$$\int \frac{\cos^2 m\psi}{\mathrm{H}(\rho\psi)} d\psi = \frac{\pi}{ab} (1 + t^m)$$

we find, on using the value of ζ_{01} got above,

$$B = -\frac{1}{ab}\left[1 + \tfrac{1}{2}t^{m-2} + \tfrac{1}{2}t^m\right] + \frac{1}{ab}\left[1 + t^m\right]$$

$$+ \frac{ab}{\pi}\left(\frac{a+b}{a-b}\right)^2 \frac{1}{m}\left[\pi \frac{d}{d\rho}\left\{\frac{e^{-m\alpha}(e^{m\alpha} + e^{-m\alpha})}{\Delta}\right\}\right.$$

$$\left. + \frac{e^{-m\alpha}(e^{m\alpha} + e^{-m\alpha})}{\Delta} \cdot \frac{\pi}{ab}(1 + t^m)\right].$$

But from (1.2) $t = e^{-2\alpha}$; $\dfrac{d\alpha}{d\rho} = \dfrac{1}{2\Delta} = \dfrac{1}{2\sqrt{\rho(\rho+q)}} = \dfrac{1}{2ab}$.

$$\therefore B = \frac{1}{2ab}(t^m - t^{m-2}) + \frac{ab}{m}\left(\frac{a+b}{a-b}\right)^2 \left[-\frac{1}{\Delta^2} \cdot \frac{2\rho + q}{2\Delta}\right.$$

$$\left. + e^{-2m\alpha}\left\{\frac{-2m}{2\Delta^2} - \frac{1}{\Delta^2}\frac{2\rho+q}{2\Delta}\right\} + \frac{1}{a^2b^2}(1 + t^m)^2\right]$$

$$= \frac{1}{2ab}(t^m - t^{m-2}) + \frac{ab}{m}\left(\frac{a+b}{a-b}\right)^2 \cdot \frac{1}{2a^2b^2}$$

$$\left[2(1+t^m)^2 - \frac{a^2+b^2}{ab}(1+t^m) - 2mt^m\right]$$

$$= \frac{t^{m-2}}{2ab}(t^2 - 1) + \frac{1}{2mab}\frac{1}{t^2}\left[2(1+t^m)^2 - \frac{a^2+b^2}{ab}(1+t^m) - 2mt^m\right]$$

$$= -\frac{t^{m-2}}{2ab}(1 - t^2) + \frac{1+t^m}{2mt^2}\frac{1}{ab}\left[2(1+t^m) - \frac{a^2+b^2}{ab}\right] - \frac{t^{m-2}}{ab}.$$

Since the equation of the given cylinder is

$$T_m \equiv \tfrac{1}{2}(1 - t^2) - \frac{1}{m}(1 + t^m) = 0,$$

we find after putting $\left(\dfrac{a^2+b^2}{ab} = 2\dfrac{1+t^2}{1-t^2}\right)$

$$B = \frac{1-t^2}{2ab}\left[\frac{1}{t^2}\left(1 - \frac{1+t^2}{1-t^2}\right)\right] - \frac{t^{m-2}}{ab}$$

$$= -\frac{1+t^{m-2}}{ab} \equiv -\frac{\pi}{\Delta} \cdot \frac{dT_m}{d\Omega},$$

as before (29.1).

§ 32. Consider now the constant A_2.

We have practically calculated this before. For, we have

$$A_2 = \text{co-efficient of } \alpha^2 \text{ in } \frac{1}{\pi}\int W \cos m\psi \, d\psi$$

$$= \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \frac{1}{\pi}\int U_2 \cos m\psi \, d\psi.$$

$$\text{But} \quad U_2 = \lim_{\substack{v=\rho \\ u\to\rho-0}} \left[\frac{1}{2}\frac{\partial}{\partial v} K\zeta'^2\, d\psi' + \zeta \frac{\partial}{\partial u}\int K\zeta'\, d\psi'\right]$$

and we have to put in this

$$\zeta = \zeta_{10}\,\alpha + \zeta_{01}\,\eta + \zeta_{20}\,\alpha^2 + \ldots\ldots$$

The co-efficient of α^2 in U_2 is therefore

$$= \text{Lim}\left[\frac{1}{2}\frac{\partial}{\partial v}\int K\zeta'^2_{10}\, d\psi' + \zeta_{10}\frac{\partial}{\partial u}\int K\zeta'_{10}\, d\psi'\right].$$

But this is precisely what we denoted before by (ζ_{10}, ζ_{10}) or (ζ_1, ζ_1), remembering that ζ_{10} is precisely the same as ζ_1 used in (12.1), viz.

$$\frac{\cos m\psi}{H(\rho\psi)}.$$

Hence
$$\pi A_2 = \int (\zeta_1, \zeta_1) \cos m\psi\, d\psi.$$

Therefore by using (15.11), we see that

$$A_2 \equiv 0 \text{ if } m \text{ is odd}$$

and that it is equal to a positive quantity if m is even.

§ 33. The calculation of A_3 is rather long and complicated though it is essentially similar to that of A_2. As however the analysis we have to go through is practically equivalent to the analysis necessary for finding a particular form of ζ_3, we may just as well start with the determination of the last named function which, we shall show, can be presented as a *finite* expression just as T_1 was (16.2). Let us do this in stages. But before we proceed with this let us find a general expression for the co-efficient of α^n in W; for, we have

$$A_3 = \text{co-efficient of } \alpha^3 \text{ in } \frac{1}{\pi}\int W \cos m\psi\, d\psi,$$

and later on when we come to discuss the convergence of the series, it will be necessary for us to find explicit expressions, or a method of obtaining such, for the co-efficient of α^n in W where

$$W = W_1\,\alpha + W_2\,\alpha^2 + W_3\,\alpha^3 + \ldots\ldots \tag{33.1}$$

For this purpose we go back to our explicit expressions for the U_n and express them in a slightly different manner which makes it possible to generalise the notation and give definite expressions for W_n.

From (7.3) we have

$$U_n = \operatorname*{Lim}_{\substack{v=\rho \\ u\to\rho-0}} \left\{ \frac{1}{n!}\frac{\partial^{n-1}}{\partial v^{n-1}}\int K\zeta'^n\, d\psi' + \ldots\ldots + \frac{\zeta^r}{(n-r)!\; r!}\frac{\partial^{n-1}}{\partial u^r\, \partial v^{n-r-1}}\int K\zeta'^{n-r}\, d\psi' + \ldots\ldots + \frac{\zeta^{n-1}}{(n-1)!\; 1!}\frac{\partial^{n-1}}{\partial u^{n-1}}\int K\zeta'\, d\psi' \right\}$$

We can express this in a slightly different form as follows:—

Consider a set of n functions of ψ, viz. $\zeta_1, \zeta_2 \ldots\ldots \xi_n$. Take any i of them and let ω_i be their product. Also let ϖ_i be the product of the other $(n-i)$. Suppose also that dashes denote (as before) that the argument is ψ' and not ψ. Put

$$n-1-i=j;\; \omega_0 = 1.$$

Form the sum

$$\sum_{i=0}^{n-i} \omega_i \frac{\partial^{n-1}}{\partial u^i\, \partial v^j}\int K\varpi'\, d\psi' \tag{33.2}$$

where the summation is extended for each value of i to all the distinct products ω_i of which the number will be ${}_nC_i$, the number of i-combinations of n things.

In this sum, where we suppose $u < v$, replace v by ρ and make

$$u \to \rho - 0.$$

Then

$$\frac{1}{n!}\sum_{i=0}^{n-1} \omega \frac{\partial^{n-1}}{\partial u^i\, \partial v^j}\int K\varpi'_i\, d\psi'$$

will be the function which we shall denote by

$$(\xi_1,\ \xi_2,\ \xi_3 \ldots\ldots \xi_n) \tag{33.3}$$

and we shall call it the parenthesis $(\xi_1,\ \xi_2 \ldots\ldots \xi_n)$.

In this notation, the expressions for U_2, U_3 ...... etc. are

$$(\zeta,\ \zeta);\ (\zeta,\ \zeta,\ \zeta);\ (\zeta,\ \zeta,\ \zeta,\ \zeta);\ \ldots\ldots$$

Properties of the Parenthesis.

(i) If we permute the ξ's in any manner whatsoever, we do not change the value of the parenthesis. This is obvious from the definition of the parenthesis.

(ii) If $\xi_1 = \phi + \chi$, then

$$(\xi_1, \xi_2 ...... \xi_n) = (\phi, \xi_2, \xi_3 ...... \xi_n) + (\chi, \xi_2, \xi_3 ...... \xi_n).$$

This is evident from the fact that any term of the left-hand side, which contains ξ_1, can at once be split up into two, one belonging to each of the parentheses on the right-hand side.

(iii) A constant factor contained in any of the functions ξ can be taken outside the parenthesis.

This is easily seen from the fact that $\omega_i \varpi'_i$ will contain that factor whatever i be and since it is independent of the integration it can be taken outside the parenthesis.

With the help of these properties of the parenthesis, we can find the expression for W_n in terms of $\zeta_1, \zeta_2 ...... \zeta_{n-1}$.

Let us remark that we have

(i) $W^* = U_2 + U_3 + ...$

$$\equiv (\zeta, \zeta) + (\zeta, \zeta, \zeta) + ...$$

(ii) U_n or $(\zeta, \zeta, ... \zeta, \zeta)_n$ has only terms of order n in ζ, ζ'.

(iii) $\zeta = \zeta_1 \alpha + \zeta_2 \alpha^2 + ...$

$$\zeta' = \zeta_1' \alpha + \zeta_2' \alpha^2 + ...$$

From (i), (ii) and (iii) it is easy to see that in order to get the co-efficient of α^n in W we need only retain the terms

$$\zeta_1 \alpha + \zeta_2 \alpha^2 + \zeta_3 \alpha^3 + ... + \zeta_{n-1} \alpha^{n-1}$$

from the series for ζ; and that in W we need only take the terms as far as

$$U_n \text{ or } (\zeta, \zeta, ... \zeta, \zeta)_n.$$

$\therefore$ W_n = co-efficient of α^n in

$$(\zeta_1 \alpha + \zeta_2 \alpha^2 + ... + \zeta_{n-1} \alpha^{n-1}, \zeta_1 \alpha + ... + \zeta_{n-1} \alpha^{n-1})$$
$$+ (\zeta_1 \alpha + ... + \zeta_{n-1} \alpha^{n-1}, \zeta_1 \alpha + ... + \zeta_{n-1} \alpha^{n-1}, \zeta_1 \alpha + ... + \zeta_{n-1} \alpha^{n-1})$$
$$+ \;...\; ... \; ... \; ... \; ... \; ... \; ... \; ...$$
$$+ (\zeta_1 \alpha + ... + \zeta_{n-1} \alpha^{n-1}, ..., \zeta_1 \alpha + \zeta_2 \alpha^2 + ... + \zeta_{n-1} \alpha^{n-1})_n.$$

Consider now the parenthesis containing S quantities each of which is the sum of r others. Say,

$$P = (\xi_1 + \xi_2 + ... + \xi_r, \eta_1 + \eta_2 + ... + \eta_r, ... ; \theta_1 + \theta_2 + ... + \theta_r)_s.$$

* See (18.5). At present we are considering the case when $\eta = 0$ or the figure of comparison is the elliptic cylinder which has the same angular velocity as the new figure.

On account of the properties of the parenthesis mentioned above P is equal to the sum of a number of other parentheses.

Thus $$P = \Sigma\,(\xi_{p_1}, \eta_{p_2}, \ldots\ldots \theta_{p_s})$$

where $p_1, p_2 \ldots\ldots p_s$ may be any of the numbers $1, 2, 3 \ldots\ldots r$ so that there are r_s parentheses in the summation.

Let us now put $\xi_{p_1} = \zeta_{p_1} \alpha^{p_1}$, $\eta_{p_2} = \zeta_{p_2} \alpha^{p_2}$, etc., so that

$$\xi_1 + \xi_2 + \ldots\ldots + \xi_r \equiv \zeta_1 \alpha + \zeta_2 \alpha^2 + \ldots\ldots + \zeta_r \alpha^r.$$

$$\eta_1 + \eta_2 + \ldots\ldots + \eta_r \equiv \zeta_1 \alpha + \zeta_2 \alpha^2 + \ldots\ldots + \zeta_r \alpha^r, \text{ etc}$$

Then

$$P = \Sigma\,(\zeta_{p_1} \alpha^{p_1}, \zeta_{p_2} \alpha^{p_2}, \ldots\ldots \zeta_{ps} \alpha^{p_s})$$

$$= \Sigma\,(\zeta_{p_1}, \zeta_{p_2}, \ldots\ldots \zeta_{p_s})\, \alpha^{p_1 + p_2 + p_3 + \ldots\ldots + p_s}.$$

Thus the co-efficient of α^{r+1} in P is

$$\Sigma\,(\zeta_{p_1}, \zeta_{p_2} \ldots\ldots \zeta_{p_s}), \qquad (33.5)$$

where $p_1, p_2, \ldots\ldots p_s$ are any of the integers $1, 2, 3 \ldots\ldots r$ such that

$$p_1 + p_2 + p_3 + \ldots\ldots + p_s = r + 1.$$

Consider now the analogous problem of finding the co-efficient of α^{r+1} in

$$(\zeta_1 \alpha + \zeta_2 \alpha^2 + \ldots\ldots + \zeta_r \alpha^r)^s$$

i.e., in the product of s factors each equal to

$$\zeta_1 \alpha + \zeta_2 \alpha^2 + \zeta_3 \alpha^3 + \ldots\ldots + \zeta_r \alpha^r.$$

To have a term with α^{r+1} we take $\zeta_{p_1} \alpha^{p_1}$ from the first factor, $\zeta_{p_2} \alpha^{p_2}$ from the second and so on, such that $p_1 + p_2 + \ldots + p_s = r + 1$ and the co-efficient of that particular term is

$$\zeta_{p_1} \zeta_{p_2} \ldots\ldots \zeta_{p_s}$$

where the above is now a product and not a parenthesis.

Thus in order to get the co-efficient of α^{r+1} in P we need only find the co-efficient of α^{r+1} in

$$(\zeta_1 \alpha + \zeta_2 \alpha^2 + \ldots\ldots + \zeta_r \alpha^r)^s$$

and in the co-efficient thus got replace any product

$$\zeta_{p_1}\ \zeta_{p_2}\ \ldots\ldots\ \zeta_{p_s}$$

by the corresponding parenthesis

$$(\zeta_{p_1},\ \zeta_{p_2}\ \ldots\ldots\ \zeta_{p_s}).$$

Thus we see that the co-efficient of α^n in W is got by finding the co-efficient of α^n in

$$(\zeta_1\ \alpha + \zeta_2\ \alpha^2 + \ldots + \zeta_{n-1}\ \alpha^{n-1})^2 + \ldots + (\zeta_1\ \alpha + \ldots + \zeta_{n-1}\ \alpha^{n-1})^n$$

and replacing any product $\zeta_{p_1}\ \zeta_{p_2}\ \ldots\ldots\ \zeta_{p_s}$ therein by the corresponding parenthesis. (33.6)

§ 34. Having thus shown how to find the explicit expressions for the W_n let us now proceed to find ζ_3. Let us note however that the expression which we shall find for ζ_3 is a particular integral of the type mentioned in § 13 though it can be proved that the most general solution is of the same type; that is to say, the most general solution for ζ_n and the particular integral T_{n-1} are both of the form

$$\frac{F(\cos\psi)}{[H(\rho\psi)]^n}$$

where F (z) is a rational integral expression in z of degree nm. But as it does not matter which solution we take for finding the increment of the angular momentum of the pear-shaped figure, we shall confine ourselves to the particular integral which is naturally the simplest.

The fundamental equation for finding ζ when the adjacent figure F is one of equilibrium is that given by (9.2), *viz.*,

$$\frac{2ab}{(a+b)^2}\ H\zeta + \frac{1}{2\pi}\int \log R^2\ (\rho\rho)\ H'\zeta'\ d\psi' = \frac{\Delta}{\pi}\ W + \text{constant}.$$

In this we substitute

$$\zeta = \zeta_1\ \alpha + \zeta_2\ \alpha^2 + \ldots\ldots$$
$$W = W_1\ \alpha + W_2\ \alpha^2 + \ldots\ldots$$

aud equate co-efficients of α^n on both sides.

This gives the equations to determine the ζ_n, namely

$$\frac{2ab}{(a+b)^2}\ H\zeta_n + \frac{1}{2\pi}\int \log R^2(\rho\rho)\ H'\zeta'_n\ d\psi' = \frac{\Delta}{\pi}\ W_n + \text{constant}. \quad (34.1)$$

As we have shown that W_n depends only on

$$\zeta_1,\ \zeta_2,\ \ldots\ldots,\ \zeta_{n-1} \quad (34.2)$$

we shall be able to solve this equation as soon as we know the functions (34.2). Hence as ζ_1 is given by

$$\zeta_1 = \frac{\cos m\psi}{H(\rho\psi)}, \tag{11.4 or 21.5}$$

we can successively find the ζ_n.

We have already found ζ_1, ζ_2 (11.4, 16.2).

To find ζ_3 we require W_3. Therefore following (33.6), we get the co-efficient of α^3 in

$$(\zeta_1\alpha + \zeta_2\alpha^2)^2 + (\zeta_1\alpha + \zeta_2\alpha^2)^3.$$

It is obviously $\qquad 2\zeta_1\zeta_2 + \zeta_1{}^3.$

Hence we have

$$\begin{aligned} W_3 &= 2(\zeta_2, \zeta_1) + (\zeta_1, \zeta_1, \zeta_1). &(34.3)\\ &= 2(T_1, T) + (T, T, T) \text{ taking only the particular integrals}\\ &= 2I_1 + I_2, \text{ say.} \end{aligned}$$

Further from (16.2) we have

$$\begin{aligned} T_1 &= \frac{1}{P}\frac{dP}{d\rho}T^2 + \frac{\Delta}{P^2}\sum_{i=1}^{m-1}\frac{P_{2i}}{T_{2i}}\frac{a'_i}{4i}\frac{\cos 2i\psi}{H(\rho\psi)}.\\ &= \frac{1}{P}\frac{dP}{d\rho}T^2 + \sum a_i\frac{\cos 2i\psi}{H(\rho\psi)}, \text{ say} \end{aligned}$$

where

$$a_i = \frac{\Delta}{P^2}\cdot\frac{P_{2i}}{T_{2i}}\frac{a'_i}{4i}. \tag{34.4}$$

Hence, remembering that by the properties of the parenthesis (§ 33) we have

$$(k_1\xi_1 + k_2\xi_2 + \ldots\ldots + k_p\xi_p, \eta) = \sum_{r=1}^{p} k_r(\xi_r, \eta)$$

we get

$$\begin{aligned} I_1 &= \frac{1}{P}\frac{dP}{d\rho}(T^2, T) + \sum a_i\left(\frac{\cos 2i\psi}{H}, T\right)\\ &= \frac{1}{P}\frac{dP}{d\rho}\left(\sum_{i=0}^{\infty}\frac{a_i\cos 2i\psi}{H}, T\right) + \sum a_i\left(\frac{\cos 2i\psi}{H}, T\right)\\ &= \frac{1}{P}\frac{dP}{d\rho}\sum_{(i)}a_i\left(\frac{\cos 2i\psi}{H}, T\right) + \sum a_i\left(\frac{\cos 2i\psi}{H}, T\right). \end{aligned}$$

Thus to find (T_1, T) or I_1, we need only evaluate

$$\left(\frac{\cos 2i\psi}{H(\rho\psi)}, T\right).$$

§ 35. Now by the definition of the parenthesis (33.3), we have

$$\left(\frac{\cos 2i\psi}{H}, T\right) = \tfrac{1}{2} \underset{\substack{v=\rho \\ u\to\rho-0}}{\mathrm{Lim}} \left\{ \begin{aligned} &\frac{\partial}{\partial v}\int KT' \frac{\cos 2i\psi'}{H(\rho\psi')}\, d\psi' \\ &+ \frac{\cos 2i\psi}{H}\frac{\partial}{\partial u}\int KT'\, d\psi' \\ &+ T\frac{\partial}{\partial u}\int K\frac{\cos 2i\psi'}{H(\rho\psi')}\, d\psi' \end{aligned} \right\}$$

$$= \tfrac{1}{2}(E_1 + E_2 + E_3), \text{ say.} \tag{35.1}$$

Consider E_2.

$$E_2 = \frac{\cos 2i\psi}{H(\rho\psi)} \mathrm{Lim} \frac{\partial}{\partial u}\int \frac{-\log R^2(uv)}{2\Delta(v)} H(v\psi') \frac{\cos m\psi'}{H(\rho\psi')}\, d\psi'.$$

As before (§ 14) since the differentiation is with respect to u, we may put $v = \rho$.

$$\text{Hence } E_2 = \frac{\cos 2i\psi}{H} \mathrm{Lim} \frac{\partial}{\partial u}\int \frac{-\log R^2(u\rho)}{2\Delta(\rho)} \cos m\psi'\, d\psi'$$

$$= \frac{\cos 2i\psi}{H} \cdot \tfrac{1}{2} \cdot \frac{2\pi}{m} Q \frac{dP}{d\rho} \cos m\psi$$

$$= \frac{\pi}{m} Q \frac{dP}{d\rho} \frac{\cos 2i\psi \cos m\psi}{H(\rho\psi)}. \tag{35.2}$$

Similarly

$$E_3 = T\,\mathrm{Lim} \frac{\partial}{\partial u}\int \frac{-\log R^2(uv)}{2\Delta(v)} H(v\psi') \frac{\cos 2i\psi'}{H(\rho\psi')}\, d\psi'$$

$$= \frac{\pi}{2i} Q_{2i} \frac{dP_{2i}}{d\rho} \frac{\cos 2i\psi \cos m\psi}{H(\rho\psi)}. \tag{35.3}$$

Consider now E_1. Using an argument similar to that employed in § 14, we get

$$E_1 = \underset{\substack{v=\rho \\ u\to\rho-0}}{\mathrm{Lim}} \frac{\partial}{\partial v}\int \frac{-\log R^2(uv)}{2\Delta(v)} H(v\psi') \frac{\cos m\psi'}{H(\rho\psi')} \cdot \frac{\cos 2i\psi'}{H(\rho\psi')}\, d\psi'$$

$$= \underset{u\to\rho-0}{\mathrm{Lim}} \int \frac{-\log R^2(u\rho)}{2\Delta(\rho)} \frac{\cos 2i\psi' \cos m\psi'}{[H(\rho\psi')]^2}\, d\psi'$$

$$+ \underset{\substack{v=\rho \\ u\to\rho-0}}{\mathrm{Lim}} \frac{\partial}{\partial v}\int \frac{-\log R^2(uv)}{2\Delta(v)} \frac{\cos 2i\psi' \cos m\psi'}{H(\rho\psi')}\, d\psi'.$$

$$\equiv \tfrac{1}{2} \underset{\substack{v=\rho \\ u\to\rho-0}}{\mathrm{Lim}} \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial \rho}\right) \int \frac{-\log R^2(uv)}{\Delta(v)} \frac{\cos 2i\psi' \cos m\psi'}{H(\rho\psi')}\, d\psi'.$$

If now we put

$$\frac{\cos 2i\psi \cos m\psi}{H(\rho\psi)} \equiv \frac{\cos 2i\psi \cos m\psi}{\rho + q \sin^2 \psi} = \sum_{l=0}^{\infty} a_{il} \cos l\psi \quad . \tag{35.4}$$

and use the equation (2.6) for $u < v$ namely:

$$\int \frac{-\log R^2(uv)}{\Delta(v)} \cos l\psi' \, d\psi' = \frac{2\pi}{l} P_l(u) Q_l(v) \text{ (using 14.2)}$$

we get

$$E_1 = \text{constant} + \tfrac{1}{2} \sum_{l=1}^{\infty} \frac{2\pi}{l} \left[\frac{dQ_l}{d\rho} P_l a_{il} - P_l Q_l \frac{da_{il}}{d\rho} \right] \times \cos l\psi. \tag{35.5}$$

Thus going back to (35.1) and using (35.2, 35.3, 35.5) we get

$$\left(\frac{\cos 2i\psi}{H}, T\right) = \pi \sum_{(l)} \cos l\psi \left\{ \begin{array}{l} \frac{P_l}{2l} \left(a_{il} \frac{dQ_l}{d\rho} - \frac{da_{il}}{d\rho} Q_l \right) \\ + \left(\frac{Q}{2m} \frac{dP}{d\rho} + \frac{Q_{2i}}{4i} \frac{dP_{2i}}{d\rho} \right) a_{il} \end{array} \right\} \tag{35.6}$$

$$= \pi \sum \overline{a_{il}} \cos l\psi, \text{ say}$$

where

$$\overline{a_{il}} = \frac{P_l}{2l} \left(a_{il} \frac{dQ_l}{d\rho} - Q_l \frac{da_{il}}{d\rho} \right) + \left(\frac{Q}{2m} \frac{dP}{d\rho} + \frac{Q_{2i}}{4i} \frac{dP_{2i}}{d\rho} \right) a_{il}. \tag{35.7}$$

As before (§ 14) using

$$\frac{Q}{m} = \frac{1}{P} \left[\frac{T_l - T}{\Delta} + \frac{1}{l} P_l Q_l \right]; \quad \frac{Q_{2i}}{2i} = \frac{1}{P_{2i}} \left[\frac{T_l - T_{2i}}{\Delta} + \frac{1}{l} P_l Q_l \right],$$

we eliminate Q, Q_{2i} and get

$$\overline{a_{il}} = \frac{P_l}{2l} \left[a_{il} \frac{dQ_l}{d\rho} - Q_l \frac{da_{il}}{d\rho} \right] + a_{il} \left\{ \begin{array}{l} \frac{1}{2P_{2i}} \frac{dP_{2i}}{d\rho} \left\{ \frac{T_l - T_{2i}}{\Delta} + \frac{1}{l} P_l Q_l \right\} \\ + \frac{1}{2P} \frac{dP}{d\rho} \left\{ \frac{T_l - T}{\Delta} + \frac{1}{l} P_l Q_l \right\} \end{array} \right\}$$

$$= \frac{P_l}{2l\, PP_{2i}} \left[a_{il} PP_{2i} \frac{dQ_l}{d\rho} - PP_{2i} Q_l \frac{da_{il}}{d\rho} + a_{il} Q_l P \frac{dP_{2i}}{d\rho} \right.$$

$$\left. + a_{il} Q_l P_{2i} \frac{dP}{d\rho} \right]$$

$$+ \left(\frac{1}{2P} \frac{dP}{d\rho} \frac{T_l - T}{\Delta} + \frac{1}{2P_{2i}} \frac{dP_{2i}}{d\rho} \frac{T_l - T_{2i}}{\Delta} \right) a_{il}.$$

Thus if

$$a'_{il} = a_{il} \frac{d(PP_{2i} Q_l)}{d\rho} - PP_{2i} Q_l \frac{da_{il}}{d\rho}, \tag{35.8}$$

we get

$$\overline{a}_{il} = \frac{P_l}{2l\, PP_{2i}} a'_{il} + \left(\frac{1}{2P} \frac{dP}{d\rho} \frac{T_l - T}{\Delta} + \frac{1}{2P_{2i}} \frac{dP_{2i}}{d\rho} \cdot \frac{T_l - T_{2i}}{\Delta} \right) a_{il}.$$

Hence from (35.6) we get

$$\left(\frac{\cos 2i\psi}{H}, T \right) = \frac{\pi}{PP_{2i}} \sum_{l=1}^{\infty} a'_{il} \frac{P_l}{2l} \cos l\psi$$

$$+ \frac{\pi}{2\Delta\, PP_{2i}} \frac{d(PP_{2i})}{d\rho} \sum T_l\, a_{il} \cos l\psi$$

$$- \frac{\pi}{2\Delta} \left(\frac{T}{P} \frac{dP}{d\rho} + \frac{T_{2i}}{P_{2i}} \frac{dP_{2i}}{d\rho} \right) \sum a_{il} \cos l\psi + \text{constant.} \tag{35.9}$$

§ 36. Consider now the formula (35.4).

Since
$$\frac{\cos 2i\psi \cos m\psi}{H(\rho\psi)} = \Sigma\, a_{il} \cos l\psi$$

we see by (15.4) that $a_{il} = 0$ if l and m are of different parity.

Hence if we put

$$l = m \pm 2r, \tag{36.1}$$

we have

$$\pi\, a_{il} = \int \frac{\cos 2i\psi \cos m\psi \cos (m \pm 2r)\psi}{H(\rho\psi)} d\psi$$

$$= \tfrac{1}{4} \int \frac{d\psi}{H(\rho\psi)} [\cos(2i \pm 2r)\psi + \cos(2i \mp 2r)\psi$$

$$+ \cos(2m + 2i \pm 2r)\psi + \cos(2m - 2i \pm 2r)\psi];$$

or
$$a_{il} = \frac{1}{2ab} \left[t^{|i \pm r|} + t^{|i \mp r|} + t^{|m+i \pm r|} + t^{|m-i \pm r|} \right].$$

Now since $i < m$ for our purposes the least of the indices of the various powers of t in a_{il} is $|i - r|$ or $|m - i - r|$.

Hence exactly as in the case of a'_{l} (§ 15), we can show that a'_{il} is identically equal to zero for l so large that $r > m - i$ or $r > i$ whichever is the larger of the two.

Hence $a'_{il} \equiv 0$, for $|l - m| > 2(m - i)$ or $2i$ whichever is the greater. Thus we can say that for $l > 2i + m$ or $l > 3m$, we shall certainly have $a'_{il} \equiv 0$.

§ 37. Let us now go back to the expansion of (T_1, T) in Fourier Series. To find the expansion of the part due to (T^2, T) let us use (35.6) combined with (35.4) which can therefore be written

$$\left(\frac{\cos 2i\psi}{H}, T\right) = \pi \sum_{l=1}^{\infty} \frac{P_l}{2l}\left(a_{il}\frac{dQ_l}{d\rho} - Q_l\frac{da_{il}}{d\rho}\right)\cos l\psi \tag{37.1}$$

$$+ \pi\left(\frac{Q}{2m}\frac{dP}{d\rho} + \frac{Q_{2i}}{4i}\frac{dP_{2i}}{d\rho}\right)\frac{\cos 2i\psi \cos m\psi}{H(\rho\psi)} + \text{constant.}$$

Let us put

$$\frac{\cos^3 m\psi}{H\rho(\psi)} \equiv \frac{\cos^3 m\psi}{\rho + q\sin^2\psi} = \sum_{(i)} b_i \cos i\psi, \tag{37.2}$$

so that

$$\pi b_i = \int_0^{2\pi} \frac{\cos^3 m\psi \cos i\psi}{H(\rho\psi)}\, d\psi.$$

Hence noting that by (14.5), (35.4)

$$\frac{\cos^2 m\psi}{H(\rho\psi)} = \sum_{(i)} a_i \cos 2i\psi\,; \quad \frac{\cos 2i\psi \cos m\psi}{H} = \sum_{(l)} a_{il}\cos l\psi$$

we get

$$\sum_{l=1}^{\infty} a_i\, a_{il} = \text{co-efficient of } \cos l\psi \text{ in } \sum_{(i)} a_i \frac{\cos 2i\psi \cos m\psi}{H}$$

$$= \text{co-efficient of } \cos l\psi \text{ in } \frac{\cos^3 m\psi}{H^2}$$

$$= \quad \text{,,} \qquad \text{,,} \text{ in} - \sum_{(l)} \frac{db_l}{d\rho}\cos l\psi$$

Hence

$$\sum_{i=1}^{\infty} a_i\, a_{il} = -\frac{db_l}{d\rho}. \tag{37.3}$$

Similarly

$$\sum_{i=1}^{\infty} a_i\frac{da_{il}}{d\rho} = \text{co-efficient of } \cos l\psi \text{ in } - \sum_{(i)} a_i \frac{\cos 2i\psi \cos m\psi}{H^2}$$

$$= \quad \text{,,} \qquad \text{,,} \quad -\frac{\cos^3 m\psi}{H^3}\,;$$

or

$$\sum_{i=1}^{\infty} a_i \frac{da_{il}}{d\rho} = -\tfrac{1}{2}\frac{d^2 b_l}{d\rho^2}. \tag{37.4}$$

Consequently

$$(T^2, T) = \Sigma\, a_i \left(\frac{\cos 2i\psi}{H}, T\right)$$

$$= \pi \sum_{(i)} a_i \left\{ \begin{array}{l} \sum_{l=1}^{\infty} \frac{P_l}{2l}\left(a_{il}\frac{dQ_l}{d\rho} - Q_l \frac{da_{il}}{dP}\right)\cos l\psi \\ + \left(\frac{Q}{2m}\frac{dP}{d\rho} + \frac{Q_{2i}}{4i}\frac{dP_{2i}}{d\rho}\right)\frac{\cos 2i\psi \cos m\psi}{H} \end{array} \right\} + \text{constant}$$

$$= \pi \sum_{l=1}^{\infty} \left[\frac{P_l}{2l}\frac{dQ_l}{d\rho}\sum_{i=1}^{\infty} a_i\, a_{il} - \frac{P_l Q_l}{2l}\sum_{i=1}^{\infty} a_i \frac{da_{il}}{d\rho}\right]\cos l\psi$$

$$+ \pi \sum_{i=1}^{\infty}\left(\frac{Q}{2m}\frac{dP}{d\rho} + \frac{Q_{2i}}{4i}\frac{dP_{2i}}{d\rho}\right)\frac{\cos m\psi}{H}\,.\, a_i \cos 2i\psi + \text{constant}.$$

Hence using (37.3), (37.4), we have

$$(T^2, T) = \text{constant} + \pi \sum_{l=1}^{\infty} \frac{P_l}{2l}\left(\tfrac{1}{2} Q_l \frac{d^2 b_l}{d\rho^2} - \frac{db_l}{d\rho}\cdot\frac{dQ_l}{d\rho}\right)\cos l\psi$$

$$+ \pi T \sum_{i=1}^{\infty} \frac{a_i}{4i} Q_{2i} \frac{dP_{2i}}{d\rho}\cos 2i\psi + \frac{\pi T^2 \cos m\psi}{2m} Q \frac{dP}{d\rho},$$

wherein we may, if we like, replace

$$T^3 \cos m\psi \equiv \frac{\cos^3 m\psi}{[H(\rho\psi)]^2} = -\sum_{l=1}^{\infty} \frac{db_l}{d\rho}\cos l\psi.$$

§ 38. We have so far considered only one part of (T_1, T). But as we do not have to transform the other part in any way, we shall leave it as it is till we consider the final form of W_3 as a Fourier Series. Let us therefore consider the expansion of (T, T, T) in Fourier Series.

Now, by the definition of the parenthesis (33.3), we have

$$(T, T, T) = \lim_{\substack{v=\rho \\ u \to \rho - 0}} \left\{ \begin{array}{l} \frac{1}{6}\frac{\partial^2}{\partial v^2}\int KT'^3 d\psi' + \frac{T}{2}\frac{\partial^2}{\partial u\, \partial v}\int KT'^3 d\psi' \\ + \frac{T^2}{2}\frac{\partial^3}{\partial u^3}\int KT'\, d\psi' \end{array} \right\}$$

$$\equiv \frac{1}{6} \text{ Lim } \frac{\partial^2}{\partial v^2} \int KT'^3 \, d\psi' +$$

$$\frac{T}{2} \text{ Lim } \frac{\partial}{\partial u} \left[\frac{\partial}{\partial v} \int KT'^2 \, d\psi' + T \frac{\partial}{\partial u} \int KT' \, d\psi' \right]$$

$$= E_1 + E_2 \text{ say.} \tag{38.1}$$

But in § 14 we have already found

$$\frac{\partial}{\partial v} \int KT'^2 \, d\psi' \; ; \; \frac{\partial}{\partial u} \int KT' \, d\psi'$$

we have

$$\frac{\partial}{\partial v} \int KT'^2 \, d\psi' = \pi \sum_{(i)} \frac{P_{2i}(u)}{2i} \left[a_i \frac{dQ_{2i}(v)}{dv} - Q_{2i}(v) \frac{da_i}{d\rho} \right] \cos 2i\psi$$

and $$\underset{v=\rho}{\text{Lim}} \frac{\partial}{\partial u} \int KT' \, d\psi' = \frac{\pi}{m} Q \frac{dP(u)}{du} \cos m\psi .$$

Hence $$E_2 \equiv \frac{T}{2} \underset{\substack{v=\rho \\ u \to \rho - 0}}{\text{Lim}} \frac{\partial}{\partial u} \left[\frac{\partial}{\partial v} \int KT'^2 \, d\psi' + T \frac{\partial}{\partial u} \int KT' \, d\psi' \right]$$

$$= \frac{T}{2} \left[\pi \sum \frac{\cos 2i\psi}{2i} \frac{dP_{2i}}{d\rho} \left(a_i \frac{dQ_{2i}}{d\rho} - Q_{2i} \frac{da_i}{d\rho} \right) + \frac{\pi Q}{m} \frac{d^2P}{d\rho^2} \frac{\cos^2 m\psi}{H} \right]$$

$$= \pi T \sum \frac{\cos 2i\psi}{4i} \frac{dP_{2i}}{d\rho} \left(a_i \frac{dQ_{2i}}{d\rho} - Q_{2i} \frac{da_i}{d\rho} \right) + \frac{\pi}{2m} Q \frac{d^2P}{d\rho^2} T^2 \cos m\psi . \tag{38.2}$$

Again

$$\text{Lim} \frac{\partial^2}{\partial v^2} \int KT'^3 \, d\psi'$$

$$= \underset{\substack{v=\rho \\ u \to \rho - 0}}{\text{Lim}} \frac{\partial^2}{\partial v^2} \int \frac{-\log R^2 (uv)}{2 \Delta (v)} H (v\psi') \frac{\cos^3 m\psi'}{[H(\rho\psi')]^3} \, d\psi' .$$

Hence following an argument similar to that used in § 14, we get this equal to

$$\text{Lim} \frac{\partial}{\partial v^2} \int \frac{-\log R^2 (uv)}{2 \Delta (v)} \frac{\cos^3 m\psi'}{[H(\rho\psi')]^2} \, d\psi'$$

$$+ 2 \text{ Lim} \frac{\partial}{\partial v} \int \frac{-\log R^2 (uv)}{2 \Delta (v)} \frac{\cos^3 m\psi'}{[H(\rho\psi')]^3} \, d\psi'$$

But from (37.2)

$$\frac{\cos^3 m\psi}{H} \equiv \frac{\cos^3 m\psi}{\rho + q\,\sin^2\psi} = \Sigma\, b_l \cos l\psi.$$

Therefore we get

$$\frac{1}{6} \operatorname{Lim} \frac{\partial^2}{\partial v^2} \int KT'^3\, d\psi'$$

$$= \frac{1}{6} \operatorname*{Lim}_{\substack{v=\rho \\ u\to\rho-0}} \left[\frac{\partial^3}{\partial v \partial \rho^2} - \frac{\partial^3}{\partial v^2 \partial \rho}\right] \int \frac{-\log R^2(uv)}{2\,\Delta(v)} \cdot \sum b_l \cos l\psi' d\psi'$$

Therefore, using the notation of (14.2) in the formulæ (2.6), we get the above equal to

$$\frac{1}{12} \operatorname{Lim} \frac{\partial^3}{\partial v \partial \rho^2} \sum \frac{2\pi}{l}\, b_l \cos l\psi \,.\, P_l(u)\, Q_l(v)$$

$$-\frac{1}{12} \operatorname{Lim} \frac{\partial^3}{\partial v^2 \partial \rho} \sum \frac{2\pi}{l}\, b_l \cos l\psi\; P_l(u)\, Q_l(v)$$

$$= \frac{\pi}{6} \sum_{(l)} \frac{P_l}{l} \left[\frac{d^2 b_l}{d\rho^2} \cdot \frac{dQ_l}{d\rho} - \frac{db_l}{d\rho} \cdot \frac{d^2 Q_l}{d\rho^2}\right] \cos l\psi. \quad (38.3)$$

Hence from (38.1), (38.2) (38.3) we get

$$(T, T, T) = \frac{\pi}{3} \sum \frac{P_l}{2l} \left(\frac{d^2 b_l}{d\rho^2} \frac{dQ_l}{d\rho} - \frac{db_l}{d\rho} \cdot \frac{d^2 Q^l}{d\rho^2} \right) \cos l\psi. \quad (38.4)$$

$$+ \pi T \sum \frac{\cos 2i\psi}{4i} \frac{dP_{2i}}{d\rho} \left(a_i \frac{dQ_{2i}}{dP} - Q_{2i} \frac{da_i}{d\rho} \right)$$

$$+ \frac{\pi}{2m}\, Q \frac{d^2 P}{d\rho^2}\, T^2 \cos m\psi + \text{constant}.$$

Hence from (37.5) and (38.4), we have

$$\frac{2}{P} \frac{dP}{d\rho} (T^2, T) + (T, T, T) \quad (38.5)$$

$$\equiv \frac{2}{P} \frac{dP}{d\rho} \left\{ \begin{aligned} &\pi \sum_{(l)} \frac{P_l}{2l} \left(\tfrac{1}{2} Q_l \frac{d^2 b_l}{d\rho^2} - \frac{db_l}{d\rho} \frac{dQ_l}{d\rho} \right) \cos l\psi \\ &+ \pi T \sum_{(i)} \frac{a_i}{4i}\, Q_{2i} \cdot \frac{dP_{2i}}{d\rho} \cos 2i\psi + \frac{\pi T^2 \cos m\psi}{2m}\, Q \frac{dP}{d\rho} \end{aligned} \right\}$$

$$+\frac{\pi}{3}\sum_{(l)}\frac{P_l}{2l}\left(\frac{d^2b_l}{d\rho^2}\cdot\frac{dQ_l}{d\rho}-\frac{db_l}{d\rho}\cdot\frac{d^2Q_l}{d\rho^2}\right)\cos l\psi$$

$$+\pi T\sum_{(i)}\frac{\cos 2i\psi}{4i}\frac{dP_{2i}}{d\rho}\left(a_i\frac{dQ_{2i}}{d\rho}-Q_{2i}\frac{da_i}{d\rho}\right)$$

$$+\frac{\pi}{2m}Q\frac{d^2P}{d\rho^2}T^2\cos m\psi+\text{constant.}$$

Let us find the co-efficient of $\cos l\psi$ under $\sum\limits_{(l)}$ in (38.5). It is

$$\frac{2\pi}{P}\cdot\frac{dP}{d\rho}\frac{P_l}{2l}\left(\tfrac{1}{2}Q_l\frac{d^2b_l}{d\rho^2}-\frac{db_l}{d\rho}\cdot\frac{d^2Q_l}{d\rho^2}\right)$$

$$+\frac{\pi}{3}\cdot\frac{P_l}{2l}\left(\frac{d^2b_l}{d\rho^2}\cdot\frac{dQ_l}{d\rho}-\frac{db_l}{d\rho}\cdot\frac{d^2Q_l}{d\rho^2}\right)$$

$$=\frac{\pi}{3P^3}\cdot\frac{P_l}{2l}\left\{\begin{aligned}&3P^2Q_l\frac{dP}{d\rho}\cdot\frac{d^2b_l}{d\rho^2}-6P^2\frac{dP}{d\rho}\frac{dQ_l}{d\rho}\cdot\frac{db_l}{d\rho}\\&+P^3\frac{d^2b_l}{d\rho^2}\cdot\frac{dQ_l}{d\rho}-P^3\frac{d^2Q_l}{d\rho^2}\cdot\frac{db_l}{d\rho}\end{aligned}\right\}$$

$$=\frac{\pi}{3P^3}\frac{P_l}{2l}\left[\frac{d^2b_l}{d\rho^2}\cdot\frac{d(P^3Q_l)}{d\rho}-\frac{db_l}{d\rho}\left\{\frac{d^2(P^3Q_l)}{d\rho^2}-Q_l\frac{d^2P^3}{d\rho^2}\right\}\right]\qquad(38.6)$$

Again if we use the equation

$$T^2\cos m\psi=\frac{\cos^3 m\psi}{H^2}=-\sum\frac{db_l}{d\rho}\cos l\psi$$

and find the co-efficient of $\cos l\psi$ in (38.5) due to the terms

$$\left[\frac{2}{P}Q\left(\frac{dP}{d\rho}\right)^2+Q\frac{d^2P}{d\rho^2}\right]\frac{\pi T^2\cos m\psi}{2m}\equiv\frac{Q}{3P^2}\frac{d^2P^3}{d\rho^2}\frac{\pi T^2\cos m\psi}{2m},$$

we see that this co-efficient is

$$-\frac{\pi}{2m}\cdot\frac{Q}{3P^2}\cdot\frac{d^2P^3}{d\rho^2}\frac{db_l}{d\rho}.\qquad(38.7)$$

Again the co-efficient of $\dfrac{\cos 2i\psi}{4i}$ under the two summations $\sum\limits_{(i)}$ in (38.5) is

$$\frac{2}{P}\cdot\frac{dP}{d\rho}\pi T a_i\frac{dP_{2i}}{d\rho}Q_{2i}+\pi T\frac{dP_{2i}}{d\rho}\left(a_i\frac{dQ_{2i}}{d\rho}-Q_{2i}\frac{da_i}{d\rho}\right)$$

$$\equiv \pi T \frac{dP_{2i}}{d\rho} \cdot \frac{1}{P^2} \left[a_i\, Q_{2i} \,.\, 2P \frac{dP}{d\rho} + a_i P^2 \frac{dQ_{2i}}{d\rho} - P_2^{\,2} Q_{2i} \frac{da_i}{d\rho} \right]$$

$$\equiv \frac{\pi T}{P^2} \cdot \frac{dP_{2i}}{d\rho} \left[a_i \frac{d(P^2 Q_{2i})}{d\rho} - P^2 Q_{2i} \frac{da_i}{d\rho} \right]$$

$$\equiv \frac{\pi T}{P^2} \frac{dP_{2i}}{d\rho} a_i'. \qquad (38.8)$$

Thus from (38.6, 38.7, 38.8) we have the following expression for (38.5)

$$\frac{\pi}{3P^3} \sum_{(l)} \frac{P_l}{2l} \left[\frac{d^2 b_l}{d\rho^2} \cdot \frac{d(P^3 Q_l)}{d\rho} - \frac{db_l}{d\rho} \cdot \frac{d^2(P^3 Q_l)}{d\rho^2} \right] \cos l\psi$$

$$+ \frac{\pi T}{P^2} \sum_{(i)} \frac{dP_{2i}}{d\rho} \frac{a'_i \cos 2i\psi}{4i} + \frac{\pi}{3\rho^3} \cdot \frac{d^2 P^3}{d\rho^2} \sum_{(i)} \left[\frac{P_l Q_l}{2l} - \frac{PQ}{2m} \right] \cos l\psi.$$

Hence if we put

$$b'_l = \frac{d^2 b_l}{d\rho^2} \cdot \frac{d(P^3 Q_l)}{d\rho} - \frac{db_l}{d\rho} \cdot \frac{d^2(P^3 Q_l)}{d\rho^2} \qquad (38.9)$$

and remember that by the definition of T_n (10.5)

$$\frac{P_l Q_l}{2l} - \frac{PQ}{2m} = \frac{T - T_l}{2\Delta}$$

we get this final expression for (38.5) :—

$$\frac{2}{P} \cdot \frac{dP}{d\rho} (T^2, T) + (T, T, T) \qquad (38.10)$$

$$= \frac{\pi}{3P^3} \sum_{(l)} \frac{b'_l}{2l} P_l \cos l\psi + \frac{\pi}{3P^3} \frac{d^2 P^3}{d\rho^2} \sum_{(l)} \frac{T - T_l}{2\Delta} \frac{db_l}{d\rho} \cos l\psi$$

$$+ \frac{\pi T}{P^2} \sum_{(i)} \frac{dP_{2i}}{d\rho} \frac{a'_i}{4i} \cos 2i\psi + \text{constant.}$$

To this we must now add

$$\sum a_i \left(\frac{\cos 2i\psi}{H}, T \right).$$

Now by (35.9) we have

$$\left(\frac{\cos 2i\psi}{H}, T\right) = \frac{\pi}{PP_{2i}} \sum_{(l)} a'_{il} \frac{P_l}{2l} \cos l\psi$$

$$+ \frac{\pi}{2\Delta\ PP_{2i}} \cdot \frac{d(PP_{2i})}{d\rho} \sum_{(l)} T_l a_{il} \cos l\psi$$

$$- \frac{\pi}{2\Delta}\left(\frac{T}{P} \cdot \frac{dP}{d\rho} + \frac{T_{2i}}{P_{2i}} \frac{dP_{2i}}{d\rho}\right) \sum_{(l)} a_{il} \cos l\psi$$

in which we shall use (35.4), *viz.*

$$T \cos 2i\psi \equiv \frac{\cos 2i\psi \cos m\psi}{H} = \sum_{(l)} a_{il} \cos l\psi.$$

Again by (34.4)

$$\alpha_i = \frac{\Delta}{P^2} \frac{P_{2i}}{T_{2i}} \frac{a'_i}{4i}$$

$$\therefore\ 2\sum \alpha_i \left(\frac{\cos 2i\psi}{H}, T\right) = \frac{2\Delta\pi}{P^3} \sum_{(i)} \frac{a'_i}{4i\, T_{2i}} \sum_{(l)} \frac{P_l}{2l} a_{il} \cos l\psi$$

$$+ \frac{\pi}{P^3} \sum_{(i)} \frac{a'_i}{4i\, T_{2i}} \frac{d(PP_{2i})}{d\rho} \sum_{(l)} T_l a'_{il} \cos l\psi.$$

$$- \frac{\pi T}{P^2} \sum_{(i)} \frac{a'_i P_{2i}}{4i\, T_{2i}} \left(\frac{T_{2i}}{P_{2i}} \frac{dP_{2i}}{d\rho} + \frac{T}{P} \cdot \frac{dP}{d\rho}\right) \cos 2i\psi. \quad (38.11)$$

From § 34 we have therefore

$$2(T_1, T) + (T, T, T) \equiv \frac{2}{P} \cdot \frac{dP}{d\rho} (\Gamma^2, T) + (T, T, T)$$

$$+ 2\sum \alpha_i \left(\frac{\cos 2i\psi}{H}, T\right) \quad (38.12)$$

$$= \frac{\pi}{3P^3} \sum_{(l)} \frac{P_l}{2l} b'_l \cos l\psi - \frac{\pi}{3P^3} \cdot \frac{d^2P^3}{d\rho^2} \sum \frac{T_l}{2\Delta} \frac{db_l}{d\rho} \cos l\psi$$

$$+ \frac{\pi}{P^3} \sum_{(i)} \frac{a'_i}{4i\, T_{2i}} \frac{d(PP_{2i})}{d\rho} \sum_{(l)} T_l a_{il} \cos l\psi$$

$$+\frac{2\pi\Delta}{P^3}\sum_{(i)}\frac{a'_i}{4i\,T_{2i}}\sum_{(l)}\frac{P_l}{2l}a'_{il}\cos l\psi$$

$$+\frac{\pi T}{P^3}\left[\frac{1}{6\Delta}\frac{d^2P^3}{d\rho^2}\sum_{(l)}\frac{db_l}{d\rho}\cos l\psi-\frac{dP}{d\rho}\sum_{(i)}\frac{a'_i\,P_{2i}}{4i\,T_{2i}}\cos 2i\psi\right]$$

$$+ \text{constant.}$$

If now we put

$$\frac{1}{6}h_l=\Delta\sum_{(i)}\frac{a'_i\,a'_{il}}{4i\,T_{2i}} \tag{38.13}$$

the third line in (38.12) can be written

$$\frac{\pi}{3P^3}\sum_{(l)}\frac{P_l}{2l}h_l\cos l\psi.$$

Hence remembering that the equation of the cylinder at the point of bifurcation is given by

$$T_m\equiv T=\tfrac{1}{2}(1-t^2)-\frac{1}{m}(1+t^m)=0$$

we get this final expression for $2(T_1, T)+(T, T, T)$ as a Fourier Series :—

$$2(T_1, T)+(T, T, T) \tag{38.14}$$

$$=\frac{\pi}{3P^3}\sum_{(l)}\frac{P_l}{2l}(b'_l+h_l)\cos l\psi-\frac{\pi}{3P^3}\frac{d^2P^3}{d\rho^2}\sum_{(l)}\frac{T_l}{2\Delta}\cdot\frac{db_l}{d\rho}\cos l\psi$$

$$+\frac{\pi}{P^3}\sum_{(i)}\frac{a_i}{4i\,T_{2i}}\frac{d(PP_{2i})}{d\rho}\sum_{(l)}T_l\,a_{il}\cos l\psi+\text{constant.}$$

Let us note that the co-efficient of $\cos m\psi$ in this is not zero. Let it be $-C'$. Now from (13.2) the equation for determining T_2 is

$$\frac{2ab}{(a+b)^2}HT_2+\frac{1}{2\pi}\int\log R^2(\rho\rho)\,H'T'_2\,d\psi'$$

$$=\frac{\Delta}{\pi}\left[2(T_1, T)+(T, T, T)+C'\cos m\psi\right]+\text{constant}$$

Hence taking the particular integral T_2 as a Fourier Series given by

$$HT_2=\sum_{(l)}{}'\,c_l\cos l\psi^*$$

* The accent indicates that the term involving $\cos m\psi$ is absent.

[library stamp]

and substituting this on the left-hand side of the above equation and comparing co-efficients of cos $l\psi$ or, which comes to the same thing, finding the various integrals of the form

$$\int HT_2 \cos l\psi \, d\psi$$

we get

$$C_l T_l = \frac{\Delta}{\pi} \left\{ \begin{aligned} &\frac{\pi}{3P^3} \frac{P_l}{2l} (b'_l + h_l) - \frac{\pi}{3P^3} \cdot \frac{d^2P^3}{d\rho^2} \frac{T_l}{2\Delta} \cdot \frac{db_l}{d\rho} \\ &+ \frac{\pi}{P^3} \sum_{(i)} \frac{a'_i a_{il}}{4i T_{2i}} \frac{d(PP_{2i})}{d\rho} T_l \quad . \end{aligned} \right\}$$

In other words

$$C_l = -\frac{1}{6P^3} \frac{d^2P^3}{d\rho^2} \frac{db_l}{d\rho} + \frac{\Delta}{3P^3} \frac{P_l}{2l} \cdot \frac{b'_l + h_l}{T_l} + \frac{\Delta}{P^3} \sum_{(i)} \frac{a'_i a_{il}}{4iT_{2i}} \frac{d(PP_{2i})}{d\rho} . \tag{38.15}$$

Hence

$$HT_2 = -\frac{1}{6P^3} \frac{d^2P^3}{d\rho^2} \sum_{(l)} \frac{db_l}{d\rho} \cos l\psi + \frac{\Delta}{3P^3} \sum_{(l)}' \frac{P_l}{2l} \cdot \frac{b'_l + h_l}{T_l} \cos l\psi + \frac{\Delta}{P^3} \cdot \sum_{(l)} \cos l\psi \sum_{(i)} \frac{a'_i a_{il}}{4i T_{2i}} \frac{d(PP_{2i})}{d\rho} .$$

But by (35.4), (37.2) we have

$$\sum_{(l)} a_{il} \cos l\psi = \frac{\cos 2i\psi \cos m\psi}{H}; \quad \sum_{(l)} \frac{db_l}{d\rho} \cos l\psi = -\frac{\cos^3 m\psi}{H^2} .$$

$$\therefore T_2 = \frac{1}{6P^3} \frac{d^2P^3}{d\rho^2} \frac{\cos^3 m\psi}{H^3} + \frac{\Delta}{P^2} \sum_{(i)} \frac{a'_i}{4i T_{2i}} \frac{d(PP_{2i})}{d\rho} \frac{\cos 2i\psi \cos m\psi}{H^2} + \frac{\Delta}{3P^3} \sum_{(l)}' \frac{P_l}{2l} \frac{b'_l + h_l}{T_l} \frac{\cos l\psi}{H} .^* \tag{38.16}$$

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* We must note here that as explained in § 13 the co-efficient of cos $m\psi$ in HT_2 is taken zero so that there are no terms with T_m in the denominator.

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NOTES AND QUESTIONS.

Notes and Questions.

"An Osculatory Fourth-difference Formula of Interpolation and its Application."*

[Mr. G. King has demonstrated an osculatory fifth-difference interpolation† and also a third-difference interpolation, and has successfully applied them in graduating the life-tables based on the Census of England of 1911. This paper is an attempt to show that a fourth-difference osculatory curve could be equally applied and that its graduating power is almost as great as that of the fifth-degree curve.]

Given a set of periodic values at equal intervals (as in the case of a Census-table wherein figures for quinquennial or decennial age-groups are given) the problem with the statistician is how best to interpolate and then to graduate the given values. It is not the purpose of this paper to consider the question of corrections to be applied to inaccurate or faulty observations, but to derive the best formula which would give graduated values for single years of age. Owing to paucity of data in the early years and in the older years of age, equal weight cannot be attached to the figures as given in the Census-table, and hence any graduation formula can be applied only to adult ages and this would depend upon the trend of the table.

The principle of osculatory method‡ is this:—Pass a series of rational algebraic curves of the same order through successive periodic values so that the successive curves intersect at given consecutive values, then pass a series of curves of a higher order through the points of intersection so that they may have the same ordinates and gradients as the original curves. These new curves are obviously continuous and must interpolate intervening ordinates. From this it follows that any number of formulæ can be constructed, and the one most suitable depends upon the circumstances of the particular table, and the facility with which the formula can be applied.

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* Extracts read by the author before the Twelfth Indian Science Congress.

† *Vide* J.I.A., Vol. XLII, pp. 240—245.

‡ *Vide* United States Life-Tables 1921, pp.

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Now, to derive a fourth-degree formula,* let u_{-2}, u_{-1}, u_0, u_1 and u_2 be the successive quinquennial values, and let curves of the third degree be passed one through u_{-2}, u_{-1}, u_0, and u_1; and another through u_{-1}, u_0, u_1, u_2. Then if u_x and v_x be ordinates of the two curves

$$u_x = u_{-2} + (x+2)\,\Delta\, u_{-2} + \frac{(x+2)(x+1)}{2!}\,\Delta^2 u_{-2}$$

$$+ \frac{(x+2)(x+1)(x)}{3!}\,\Delta^3 u_{-2} \qquad (1)$$

$$v_x = u_{-1} + (x+1)\,\Delta\, u_{-1} + \frac{(x+1)(x)}{2!}\,\Delta^2 u_{-1}$$

$$+ \frac{(x+1)(x)(x-1)}{3!}\,\Delta^3 u_{-1} \qquad (2)$$

where Δ is the symbol of finite differences.

But since

$$u_{-1} = u_{-2} + \Delta\, u_{-2};\quad \Delta^p u_{-1} = \Delta^p u_{-2} + \Delta^{p+1} u_{-2}$$

it follows that

$$v_x = u_x + \frac{(x+1)(x)(x-1)}{3!}\,\Delta^4 u_{-2} \qquad (3)$$

Let y_x be the required osculatory curve of the fourth order passing through u_{-1}, u_0 so that at u_{-1} the curve has the same gradient as u_x and at u_0 the same gradient as v_x; so that at these points the required curve has the same ordinate and gradient as u_x and v_x.

Now let $y_x = u_x + \psi_x$, where ψ_x is of the fourth degree

then $\quad y_{-1} = u_{-1},\ y'_{-1} = u'_{-1};\ y_0 = v_0,\ y'_0 = v'_0. \qquad (4)$

From the first two equations it easily follows that:

$$\psi_{-1} = 0;\ \psi'_{-1} = 0;$$

that is, ψ_x must contain $(x+1)^2$ as a factor.

Again from (3)

$$y_0 - v_0 = \psi_0$$

$$y'_0 - v'_0 = \psi'_0 + \tfrac{1}{6}\Delta^4 u_{-2}.$$

Therefore from the last two equations of (4) we have

$$\psi_0 = 0;\ \psi'_0 = -\tfrac{1}{6}\Delta^4 u_{-2}.$$

Hence since $\psi_0 = 0$ $\psi(x)$ contains the factor x.

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* *Vide* J.I.A., Vol. XLII, pp. 394—397.

Let us now assume that

$$\psi(x) = (x+1)^2\, x(x+l)\ \Delta^4 u_{-2}$$

which gives

$$\psi'_0 = {}_{x=0}\{4x^3 + (2+l)\,3x^2 + (2l+1)\,2x + l\}\ \Delta^4 u_{-2}$$

This shows that $l = -\frac{1}{6}$.

Hence $$\psi_x = \frac{x(x+1)^2(6x-1)}{3\,!}\ \Delta^4 u_{-2}.$$

Therefore the fourth-degree interpolation formula takes the form :—

$$y_x = u_{-2} + (x+2)\,\Delta u_{-2} + \frac{(x+2)(x+1)}{2\,!}\,\Delta^2 u_{-2}$$

$$+ \frac{(x+2)(x+1)(x)}{3\,!}\,\Delta^3 u_{-2} + \frac{x(x+1)^2(6x-1)}{3\,!}\,\Delta^4 u_{-2}.$$

Now the practical application of the formula and the facility with which we can apply it depends upon the forms of sub-divided differences. By differencing the last equation for the interval $\frac{1}{t}$ (with the usual notation δ to indicate sub-divided difference, and Δ the major difference)

$$\delta y_{-1} = \frac{\Delta u_{-2}}{t} + \frac{(t+1)}{2\,!}\,\frac{\Delta^2 u_{-2}}{t^2} - \frac{(t^3-1)}{3\,!}\,\frac{\Delta^3 u_{-2}}{t^3}$$

$$+ \frac{7t^2 - 13t + 6}{3\,!}\,\frac{\Delta^4 u_{-2}}{t^4}$$

$$\delta^2 y_{-1} = \frac{\Delta^2 u_{-2}}{t^2} + \frac{\Delta^3 u_{-2}}{t^3} + \frac{7t^2 - 39t + 42}{3}\,\frac{\Delta^4 u_{-2}}{t^4}$$

$$\delta^3 y_{-1} = \frac{\Delta^3 u_{-2}}{t^3} - (13t - 36)\,\frac{\Delta^4 u_{-2}}{t^4}$$

$$\delta^4 y_{-1} = 24\,\frac{\Delta^4 u_{-2}}{t^4}.$$

Now taking $t = 5$ (the usual range of the Census-table), these sub-divided differences reduce to :—

$$\delta y_{-1} = \frac{\Delta u_{-2}}{5} + \frac{3\,\Delta^2 u_{-2}}{5^2} - \frac{4\,\Delta^3 u_{-2}}{5^3} + \frac{58}{3}\cdot\frac{\Delta^4 u_{-2}}{5^4}$$

$$\delta^2 y_{-1} = \frac{\Delta^2 u_{-2}}{5^2} + \frac{\Delta^3 u_{-2}}{5^3} + \frac{22}{3}\,\frac{\Delta^4 u_{-3}}{5^4}$$

$$\delta^3 y_{-1} = \frac{\Delta^3 u_{-2}}{5^3} - 29\cdot\frac{\Delta^4 u_{-2}}{5^4}$$

$$\delta^4 y_{-1} = 24\cdot\frac{\Delta^4 u_{-3}}{5^4}.$$

The interpolated values of y for each interval are then formed by the repeated additon of the sub-divided differences.

The formula of the fourth order gives as smooth a result as the fifth and has the advantage of being less laborious. Though the co-efficients involved in the sub-divided differences are not so neat as in the similar formulæ for the third and the fifth, it is found from experience that the labour involved is not very great and we have the decided advantage of a better graduation It may be noted that so long as the observations have the same weight any number of formulæ may be evolved, and the most successful, *i.e.* the one which gives fairly close representation of original facts, must depend upon how the original table is compiled. It is found in graduating the Indian Census that the rates of mortality after graduation by means of the fourth degree formula give indeed excellent results.

M. VAIDYANATHAN.

To find the factors of N = 35821.

$N = 189^2 + 100.$

If x be a factor (prime or not) of N, it is also factor of $(189-x)^2+100$; the latter number being obviously divisible by 4, we write

$$(189 - x)^2 + 100 = 4kx.$$

(showing that every odd factor of x and of k is of the form $(4p + 1)$).

Rearranging $\quad x^2 - 2(2k + 189)x + N = 0;$

or $\quad [x - (2k + 189)]^2 = (2k + 189)^2 - N.$

Hence $\quad (2k + 189)^2 - (189^2 + 100)$

is a quadratic residue for all moduli.

Taking 32 to be the modulus,

$$(2k - 3)^2 - 13 \equiv \text{a } q.r. \equiv 0\ 4, 16, 1, 9, 17 \text{ or } 25$$

or $\quad (2k - 3)^2 \equiv 13, 17, 29, 14, 22, 30 \text{ or } 38.$

The only admissible value is 17

$\therefore \quad (2k - 3)^2 \equiv 17 \equiv 49 \pmod{32};$

or $\quad 2k - 3 \equiv 7 \text{ or } 9 \pmod{16};$ $\quad$ or $\quad k \equiv 5 \text{ or } 6 \pmod{8}.$

Also $\quad k = 8q + 6 = 2(4q + 3)$ is inadmissible.

$$\therefore \quad k = 8\lambda + 5.$$

Next take 9 to be the modulus, we get

$$4k^2 - 1 \equiv 0 \text{ or } 1, 4, 7,$$

or
$$4k^2 \equiv 1 \pmod{9}$$

and
$$k^2 \equiv 16;$$

$\therefore$
$$k \equiv 4 \text{ or } 5 \pmod{9}.$$

Combining the two we get

$$k \equiv 72\,\lambda_1 + 5 \text{ or } 72\,\lambda_1 + 13 \equiv 5, 13, 77, 86, 149, 157, 221, 229, 293 \text{ or } 301 \pmod{360}.$$

The modulus 25, gives

$$(2k - 11)^2 \equiv 11^2 + (0.5p \pm 1) \pmod{25}$$

that is, $k \equiv 0$ or $11 \pmod{25}$; or $(2k - 1)^2 \equiv 0\,(, 5)$ and $k \equiv 3(, 5)$

The only possible value of k up to 292 is $k = 13$. It is easily seen that as x decreases, k increases; and that when $x = 29$, $k < 222$. Thus if $k = 13$ does not give a factor, there is no factor $\geqslant 29$. The possible prime factors below 29 are 5, 13, 17; and none of them is a factor.

$k = 13$, gives

$$(2k + 189)^2 - N = 215^2 - N = 46325 - 35821 = 10404 = 102^2.$$

$$\therefore \; N = 215^2 - 102^2 = 113.317.$$

BALAK RAM.

Solutions.

Question 1252.

(K. J. SANJANA, M.A.):—Prove that for real integral values of x and y the equality $6x^2+2 = y^3$ is impossible except in the single case $x = 1$ and $y = 2$. Examine if the equality holds for any other real rational values of x and y.

Solution by S. Audinaraniah, I. Todadri Iyengar and S. D. Chowla.

It is well-known that the equality $x^3 + y^3 = z^3$ has no solution in integers. From this it is easy to deduce that it has no rational solution either. Now we have identically $6x^2 + 2 = (1 + x)^3 + (1 - x)^3$. The equality given us becomes $(1 + x)^3 + (1 - x)^3 = y^3$.

Hence it does not possess any rational or integral solutions except those which make any of the members vanish.

Hence the only solutions are $x = \pm 1$, $y = 2$.

Question 1300.

(M. G. IMAMDAR):—Prove that the area of a triangle can be got from the formula

$$\Delta = 4\{(\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R)\}^{\frac{1}{2}},$$

where $2\sigma = p + q + r + R$ and p, q, r are the pependiculars from the circumcentre on the sides of the triangle and R is the circumradius.

Comment by E. H. Neville.

The geometrical solution of this question given on p. *206* of the section of Vol. 15 devoted to Notes and Questions needs a slight modification to make clear the essence of the problem and the necessity for the qualification that the triangle is to be accute-angled. Suppose that instead of constructing the right-angled triangle DEF′ of Mr. Satyanarayana's figure we construct a similar triangle on BA; then if S is the circumcentre, the hypotenuse is not merely parallel to AS but coincides with AS, and the third vertex is the point in which AS cuts the circumcircle again. Let us follow up this clue.

If AA′, BB′, CC′ are three diameters of a circle, and if H is the orthocentre of ABC, then A′B and CH are parallel, since they are both perpendicular to AB, and A′C and BH are parallel, since they are both perpendicular to AC. Hence BA′CH is a parallelogram, and the triangles CBA′, BCH have the same area. If then H is inside ABC, that is, if ABC is acute-angled, the sum of the areas of the three triangles CBA′, ACB′, BAC′ is the area of ABC, and the area of ABC is half that of the hexagon BA′CB′AC′.

On the other hand, each of the diameters AA′, BB′, CC′ cuts the hexagon into two congruent quadrilaterals, and in the rectangle BCB′C′ the length of the sides CB′, C′B is twice the distance of BC from the centre S. That is, the hexagon can be cut, in three different ways, into two halves each of which is a cyclic quadrilateral with the sides $2p$, $2q$, $2r$, 2R, and therefore with the area

$$4\{(\sigma-p)(\sigma-q)(\sigma-r)(\sigma-R)\}^{\frac{1}{2}}$$

Question 1304—(*Corrected*).

(N. R. JAYA RAO, M.A.) :—

If
$$\frac{\pi}{2}\cdot\frac{1.3.5.7\ldots\ldots(2k-1)}{2.4.6.8\ldots\ldots 2k}=f(k),$$

then show that
$$\int_0^\infty \frac{\sin^{2k+1}x}{x}\,dx=f(k).$$

and
$$\int_0^\infty \frac{\sin^{2k}x}{x^2}\,dx=f(k-1).$$

Solution by Martyn Thomas.

Lemma:
$$\int_0^\infty \frac{\sin rx}{x}\,dx=\frac{\pi}{2},\ (r>0).$$

Let $y=\cos x+i\sin x$; then $\frac{1}{y}=\cos x-i\sin x$;

so that $y+\frac{1}{y}=2\cos x$ and $y-\frac{1}{y}=2i\sin x$.

$$\therefore\ \left(2i\sin x\right)^{2k+1}=\left(y-\frac{1}{y}\right)^{2k+1}=\left(y^{2k+1}-\frac{1}{y^{2k+1}}\right)-C_1$$
$$\left(y^{2k-1}-\frac{1}{y^{2k-1}}\right)+\ldots\ldots\ (-1)^k C_k\left(y-\frac{1}{y}\right)$$
$$=2i\,[\sin(2k+1)x-C_1\sin(2k-1)x$$
$$+\ldots\ldots+(-1)^k.C_k\sin x]$$

$$\therefore \sin^{2k+1} x = \frac{(-1)^k}{2^{2k}} \Big[\sin(2k+1)x - C_1 \sin(2k-1)x + \ldots\ldots + (-1)^k . C_k \sin x \Big].$$

$$\therefore \int_0^\infty \frac{\sin^{2k+1} x}{x} dx = \frac{(-1)^k}{2^{2k}} . \frac{\pi}{2} [1 - C_1 + C_2 - \ldots\ldots + (-1)^k . C_k], \text{ by Lemma}$$

$$= \frac{(-1)^k}{2^{2k}} . \frac{\pi}{2} [(-1)^k \times \text{co-efficient of } z^k \text{ in } (1+z)^{2k+1} \times (1+z)^{-1}]$$

$$= \frac{\pi}{2} . \frac{2k\,!}{(2^k\, k!)\,(2^k\, k!)}$$

$$= \frac{\pi}{2} . \frac{1.3.5 \ldots\ldots (2k-1)}{2.4.6 \ldots\ldots 2k} = f(k).$$

Again,

$$\int_0^\infty \frac{\sin^{2k} x}{x^2} dx = \Big[-\frac{\sin^{2k} x}{x} \Big]_0^\infty + 2k \int_0^\infty \frac{\sin^{2k-1} x \cos x}{x} dx$$

$$= 2k \int_0^\infty \frac{\sin^{2k-1} x \cos x}{x} dx$$

Again, $(2i \sin x)^{2k} = \left(y - \frac{1}{y} \right)^{2k}$

$$= \left(y^{2k} + \frac{1}{y^{2k}} \right) - {}_{2k}C_1 \left(y^{2k-2} + \frac{1}{y^{2k-2}} \right) + \ldots\ldots (-1)^{k-1} . {}_{2k}C_{k-1} \left(y^2 + \frac{1}{y^2} \right) + (-1)^k . {}_{2k}C_k.$$

$$\therefore 2^{2k} (-1)^k \sin^{2k} x = 2 [\cos 2kx - {}_{2k}C_1 \cos(2k-2)x + \ldots\ldots (-1)^{k-1} . {}_{2k}C_{k-1} \cos 2x] + (-1)^k {}_{2k}C_k.$$

$\therefore$ By differentiation,

$$2^{2k} (-1)^k \cdot 2k \sin^{2k-1} x \cos x = -2 [2k \sin 2kx - (2k-2) . {}_{2k}C_1 \times \sin(2k-2)x \ldots\ldots (-1)^{k-1} . 2 . {}_{2k}C_{k-1} \sin 2x]$$

$$\therefore 2k \sin^{2k-1} x \cos x = \frac{(-1)^{k-1}}{2^{2k-1}} \Big\{ 2k \sin 2kx - (2k-2) . {}_{2k}C_1 \sin(2k-2)x + \ldots\ldots (-1)^{k-1} . 2 . {}_{2k}C_{k-1} \sin 2x \Big\}$$

$$\therefore \quad \int_0^\infty \frac{\sin^{2k} x}{x^2}\, dx \equiv \int_0^\infty \frac{2k \sin^{2k-1} x \cos x}{x}\, dx$$

$$= \frac{(-1)^{k-1}}{2^{2k-1}} \cdot \frac{\pi}{2} \left\{ 2k - (2k-2) \cdot {}_{2k}C_1 \ldots\ldots + (-1)^{k-1} . 2 \cdot {}_{2k}C_{k-1} \right\},$$

$$= \frac{(-1)^{k-1}}{2^{2k-1}} \cdot \frac{\pi}{2} \cdot (-1)^{k-1}\, 2 \left\{ {}_{2k}C_{k-1} - 2 \cdot {}_{2k}C_{k-2} + 3 \cdot {}_{2k}C_{k-3} \right.$$

$$\left. \ldots\ldots (-1)^{k-1} . k \cdot {}_{2k}C_0 \right\}$$

$$= \frac{\pi}{2^{2k-1}} \left\{ \text{co-efficient of } z^{k-1} \text{ in } (1+z)^{2k} \times (1+z)^{-2} \right\}$$

$$= \frac{\pi}{2^{2k-1}} \cdot {}_{2k-2}C_{k-1} = \frac{\pi}{2^{2k-1}} \frac{2k-2\,!}{k-1\,!\;k-1\,!}$$

$$= \frac{\pi}{2} \cdot \frac{2k-2\,!}{(2^{k-1}\, k-1\,!)\,(2^{k-1}\, k-1\,!)}$$

$$= \frac{\pi}{2} \cdot \frac{1.3.5 \ldots\ldots (2k-3)}{2.4.6 \ldots\ldots (2k-2)}$$

$$= f(k-1).$$

Question 1319.

(B. B. BAGI) :—ABC is a spherical triangle. Any point P on the sphere is joined to A, B, C and the cosines of these arcs are denoted by (a_1, b_1, c_1); similarly (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4) denote the cosines of the arcs joining three other points Q, R, S to the vertices of A, B, C. Prove that the four points P, Q, R, S lie on a small circle, if

$$\begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \\ a_4 & b_4 & c_4 & 1 \end{vmatrix} = 0.$$

Solution by R. Srinivasan and C. Ranganatha Iyengar.

Let O be the centre of the small circle on which P, Q, R, S lie; and $2n_1$, $2n_2$, $2n_3$, $2n$, be the sines of the Δs OBC, OCA, OAB, ABC. Then we have (*vide*: Todhunter and Leathem's *Spherical Trignometry*, § 165).

$$n_1 a_1 + n_2 b_1 + n_3 c_1 - n \cos \mathrm{PO} = 0,$$
$$n_1 a_2 + n_2 b_2 + n_3 c_2 - n \cos \mathrm{QO} = 0,$$
$$n_1 a_3 + n_2 b_3 + n_3 c_3 - n \cos \mathrm{RO} = 0,$$
$$n_1 a_4 + n_2 b_4 + n_3 c_4 - n \cos \mathrm{SO} = 0.$$

Remembering that PO = QO = RO = SO, and eliminating n_1, n_2, n_3, n_4, n, we get the result.

[*Other Solutions by Martyn Thomas and M. L. Chandratreya.*]

Question 1342.

(V. RAMASWAMI AIYER) :—Prove that

$$\begin{vmatrix} 0 & x & x' & 0 & a & a' \\ 0 & x' & x & 0 & a' & a \\ y' & 0 & y & b' & 0 & b \\ y & 0 & y' & b & 0 & b' \\ z & z' & 0 & c & c' & 0 \\ z' & z & 0 & c' & c & 0 \end{vmatrix} = \begin{vmatrix} x^2 - x'^2 & y^2 - y'^2 & z^2 - z'^2 \\ a^2 - a'^2 & b^2 - b'^2 & c^2 - c'^2 \\ ax - a'x' & by - b'y' & cz - c'z' \end{vmatrix}$$

Solution by S. Audinarayanan.

Expanding the left-side in terms of the minors of the first two rows, we have

$$(x^2 - x'^2)\begin{vmatrix} y' & b' & 0 & b \\ y & b & 0 & b' \\ z & c & c' & 0 \\ z' & c' & c & 0 \end{vmatrix} + (a'x - ax')\begin{vmatrix} y' & y & b' & b \\ y & y' & b & b' \\ z & 0 & c & 0 \\ z' & 0 & c' & 0 \end{vmatrix}$$

$$-(ax - a'x')\begin{vmatrix} y' & y & b' & 0 \\ y & y' & b & 0 \\ z & 0 & c & c' \\ z' & 0 & c' & c \end{vmatrix} + (ax - a'x')\begin{vmatrix} y' & 0 & b' & b \\ y & 0 & b & b' \\ z & z' & c & 0 \\ z' & z & c' & 0 \end{vmatrix}$$

$$+(ax' - a'x)\begin{vmatrix} y' & 0 & b' & 0 \\ y & 0 & b & 0 \\ z & z & c & c' \\ z' & z' & c' & c \end{vmatrix} + (a^2 - a'^2)\begin{vmatrix} y' & 0 & y & b' \\ y & 0 & y' & b \\ z & z' & 0 & c \\ z' & z & 0 & c' \end{vmatrix}$$

$$= (x^2 - x'^2)\begin{vmatrix} b^2 - b'^2, & c^2 - c'^2 \\ by - b'y', & cz - c'z' \end{vmatrix} + (a'x - ax')(by' - b'y)(cz' - c'z)$$

$$-(a'x - ax')(by' - b'y)(cz' - c'z)$$

$$+(ax - a'x')\,[(cz - c'z')(by - b'y') + (c^2 - c'^2)(y^2 - y'^2)]$$

$$-(ax - a'x')\,[(cz - c'z')(by - b'y') + (b^2 - b'^2)(z^2 - z'^2)]$$

$$-(a^2-a'^2)\begin{vmatrix} y^2-y'^2, & z^2-z'^2 \\ by-b'y', & cz-c'z' \end{vmatrix}$$

$$=\begin{vmatrix} x^2-x'^2 & y^2-y'^2 & z^2-z'^2 \\ a^2-a'^2 & b^2-b'^2 & c^2-c'^2 \\ ax-a'x' & by-b'y' & cz-c'z'. \end{vmatrix}$$

Other Solutions by Martyn Thomas and A. Muniswami].

Question 1362.

(S. Narayana Aiyar) :—Show that

$$\frac{1-\operatorname{dn} u \operatorname{dn} v \operatorname{dn}(u+v)}{1-\operatorname{cn} u \operatorname{cn} v \operatorname{cn}(u+v)}=k^2.$$

Solution by Hans R. Gupta, Ram Behari and Vidya Chandra.

Denoting $\operatorname{sn} u$ by s_1, etc. and $\operatorname{sn} v$ by s_2, etc., we have

$$\operatorname{dn}(u+v)=(d_1d_2-k^2s_1s_2c_1c_2)\div(1-k^2s_1^2s_2^2)$$

and
$$\operatorname{cn}(u+v)=(c_1c_2-s_1s_2d_1d_2)\div(1-k^2s_1^2s_2^2);$$

so that

$$\frac{1-\operatorname{dn} u \operatorname{dn} v \operatorname{dn}(u+v)}{1-\operatorname{cn} u \operatorname{cn} v \operatorname{cn}(u+v)}=\frac{1-k^2s_1^2s_2^2-d_1^2d_2^2+k^2s_1s_2c_1c_2d_1d_2}{1-k^2s_1^2s_2^2-c_1^2c_2^2+s_1s_2c_1c_2d_1d_2}.$$

Now

$$\begin{aligned}1-k^2s_1^2s_2^2-d_1^2d_2^2 &= 1-k^2s_1^2s_2^2-(1-k^2s_1^2)(1-k^2s_2^2)\\ &= k^2(s_1^2+s_2^2-s_1^2s_2^2-k^2s_1^2s_2^2)\\ &= k^2\{s_1^2+s_2^2-(1-c_1^2)(1-c_2^2)-k^2s_1^2s_2^2\}\\ &= k^2\{s_1^2+c_1^2+s_2^2+c_2^2-1-c_1^2c_2^2-k^2s_1^2s_2^2\}\\ &= k^2\{1-c_1^2c_2^2-k^2s_1^2s_2^2\}.\end{aligned}$$

Hence the given expression is equal to k^2.

Question 1365.

(P. L. SRIVASTAVA):—A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal, and rests with its lower end on the quadrant of the curve which is farthest from the focus. Find its position of equilibrium and show that its length must be at least $\frac{3a}{4}+\frac{9}{4}\sqrt{1+8e^2}$, where $2a$ is the major-axis and e the eccentricity.

Solution by K. V. Srikantesvaran, M. R. Chandratreya, N. N. Bose, D. K. Sen and Ram Behari.

[The question is given in Routh's *Statics* as Ex. 5, on p. 147].

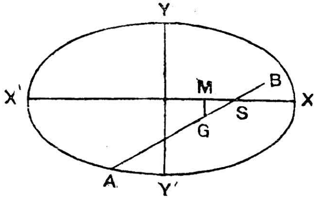
If y be the ordinate of G, the middle point of the rod, then for the position of equilibrium, we have by the principle of virtual work

$$\frac{dy}{dr}=0, \text{ where } r=\text{SA}.$$

Now $$y=\text{SG}\sin \text{G}\hat{\text{S}}\text{M}=\left(r-\frac{l}{2}\right)\frac{\sqrt{1-e^2}}{re}\sqrt{2ar-r^2-a^2\cdot\overline{1-e^2}}$$

since $$(1-e\cos \text{G}\hat{\text{S}}\text{M})=\frac{a(1-e^2)}{r},$$

l representing the length of the rod.



Differentiating with respect to r,

$$(2ar-r^2-a^2.\overline{1-e^2})^{-\frac{1}{2}}(a-r)-\frac{l(a-r)}{2}(2ar-r^2-a^2\,\overline{1-e^2})^{-\frac{1}{2}}$$
$$+\frac{l}{2}\cdot\{2ar-r^2-a^2(1-e^2)\}^{\frac{1}{2}}=0.$$

Simplifying and arranging in powers of l, we have

$$2r^3-2ar^2-ral+a^2l(1-e^2)=0.$$

The roots of this cubic give the values of SA for the positions of equilibrium.

In order that there be a position of equilibrium in which A lies in the quadrant X′AY′ of the curve, SA or r must be less than l and greater than a (the value of SY′). If the equation in r given above has a root between l and a, the expressions

$$2a^3 - 2a^3 - a^2 l + a^2 l\,(1 - e^2);\ 2l^3 - 2al^2 - al^2 + a^2 l\,(1 - e^2)$$

must be of opposite signs. The first of these is negative being equal to $-a^2e^2l$ Hence the second must be positive. That is,

$$\left(l - \frac{3a}{4} + \frac{a}{4}\sqrt{1+8e^2}\right)\left(l - \frac{3a}{4} - \frac{a}{4}\sqrt{1+8e^2}\right) \text{ must be } + ve.$$

$$\therefore \qquad l > \frac{3a}{4} + \frac{a}{4}\sqrt{1+8e^2}.$$

The geometrical meaning of the condition is that the intersection of the normal to the ellipse at A and the perpendicular to the rod at S should lie to the right of the mid-point of SA.

The reactions at A, S and the weight of the rod at G are the only three forces acting on the rod. Hence the above condition.

The equations may be obtained in terms of the cosine of the inclination of the rod to the major axis. In that case, the resulting cubic in $\cos\theta$ would be

$$\cos^3\theta - \frac{2}{e}\cos^2\theta + \frac{l - 2a.1 - e^2}{le^2}\cos\theta + \frac{2a}{le}(1 - e^2) = 0;$$

and this equation in $\cos\theta$ must have a root between e and $\frac{(l-a)\,(1-e^2)}{le}$ corresponding to the values a and l of SY′ and SA respectively.

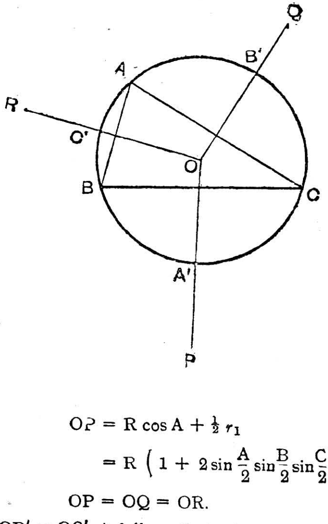
Question 1371.

(B. B. BAGI) :—If at the middle points of an acute-angled triangle ABC, perpendiculars to the sides are erected of lengths $\frac{1}{2}r_1$, $\frac{1}{2}r_2$, $\frac{1}{2}r_3$, all drawn outwards, then the triangle whose vertices are the extremities of these perpendiculars is similar to the ex-central triangle.

Solution by V. A. Apte.

Let A′, B′, C′ be the mid.-points of the arcs BC, CA, AB of the circum-circle of the ΔABC. Then we know that Δ A′B′C′ is similar to the Δ $I_1I_2I_3$.

Now if P, Q, R be the angular points of the proposed triangle so that OP = R cos A + $\frac{1}{2}r_1$, etc.



We have

$$OP = R\cos A + \tfrac{1}{2}r_1$$
$$= R\left(1 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

$\therefore$ OP = OQ = OR.

Since OA′ = OB′ = OC′, it follows that PQR is similar to A′B′C′.

Hence PQR is similar to $I_1I_2I_3$.

Additional Solutions by Ramlal Chitkara and K. C. Mowleswar.

Questions for Solution.

1400. (R. GOPALASWAMI):—Find the relation between the mutual invariants of three conics, which are such that a conic can be drawn to have double contact with all of them.

Show that any three conics circumscribing a quadrangle, or inscribed in a quadrilateral, satisfy this condition and that their three chords of contact with the enveloping conic form a self-conjugate triangle for the latter.

1401. (S. D. CHOWLA):—With the usual notation for Bernoulli's numbers, prove that

(1) $2\{ {}^{2p}C_2\, B_2 - {}^{2p}C_4\, B_4 + {}^{2p}C_6\, B_6 \dots (-1)^p\ {}^{2p}C_{2p-2}\, B_{2p-2} \}$

is always an integer.

(2) $$\frac{B_2}{1.2} - {}^{2p}C_2\, \frac{B_4}{3.4} + {}^{2p}C_4\, \frac{B_6}{5.6} \dots\dots + (-1)^{p-1}\ {}^{2p}C_{2p-2}\, \frac{B_{2p}}{(2p-1)(2p)} = \frac{1}{2p+2} - \frac{1}{2(2p+1)}.$$

1402. (S. D. CHOWLA):—Prove that

$$\Sigma \{ \phi(n_1) \,.\, \lambda(n_2) \} = d(n)$$

where $d(n)$ denotes the sum of the divisors of n, $\phi(n)$ the number of numbers less than n and prime to it, $\lambda(n)$ the number of divisors of n; n_1 and n_2 are conjugate divisors of n, and the summation extends over all the divisors of n.

Example: $d(15) = 24$, and

$$\lambda(1)\, \phi(15) + \phi(1)\, \lambda(15) + \lambda(3)\, \phi(5) + \lambda(5)\, \phi(3) = 24$$

1403. (A. NARASINGA RAO):—Find the envelope of the axis of a conic passing through four fixed points.

Discuss the specialisations brought about by the points

(i) forming an orthocentric system,

(ii) lying on a circle.

1404. (A. NARASINGA RAO) :—Through the in- and ex-centres of a triangle ABC perpendiculars are erected to the plane on both sides of it of lengths equal to the radii of the corresponding circles. Denoting by accented letters points on one side of the plane, prove that the two tetrahedra $(i_1 i_2 i_3 i')$ and $(i_1' i_2' i_3' i)$ are in quadruple perspective, their four centres of perspective being A, B, C and a certain point at infinity.

Conversely, given two desmic tetrahedra which are the reflexions, each of the other in a certain plane, discuss whether their vertices correspond, in the manner stated above, to the in- and ex-circles of a triangle in that plane ?

1405. (S. NARAYANA AIYAR, M.A.) :—If P and Q denote the expressions

$$\sin\left[\tan^{-1}\{\tan\theta\sqrt{1-k^2\sin^2\phi}\}+\tan^{-1}\{\tan\phi\sqrt{1-k^2\sin^2\theta}\}\right]$$

$$\sin\left[\tan^{-1}\frac{k\sin\theta\cos\phi}{\sqrt{1-k^2\sin^2\theta}}+\tan^{-1}\frac{k\sin\phi\cos\theta}{\sqrt{1-k^2\sin^2\phi}}\right],$$

show that the ratio $\frac{P}{Q}$ is independent of θ and ϕ and is equal to k.

1406. (G. S. MAHAJANI) :—Sum the series

$$(1)\quad \sum_{p,\,q,\,r}\frac{p^4}{(p^2+q^2+r^2)^{9/2}}$$

$$(2)\quad \sum_{p,\,q,\,r}\frac{p^2q^2}{(p^2+q^2+r^2)^{9/2}}$$

where p, q, r take all integral values from $-\infty$ to $+\infty$, in such a way that in any one term they are *all* odd, or *all* even.

[These series turn up in dealing with magnetic properties of iron crystals, whose structure is "body-centered cubic." An approximate evaluation at least is required].

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LIST OF BOOKS AND JOURNALS RECEIVED IN THE LIBRARY

(during July and August 1925.)

Journals.

1 American Journal of Mathematics, Vol. 47, No. 2 (2 copies).
2 American Mathematical Monthly, Vol. 32, Nos. 1, 5, 6 (2 copies each) and Vol. 25, No. 1.
3 Annales Scientifiques de l'Ecole Normale Superieure, 1925, Nos. 3, 4, 5 and 6.
4 Annals of Mathematics, Vol. 26, No. 3.
5 Annurio, No. 62.
6 Astrophysical Journal, Vol. 61, No. 4.
7 Bulletin des Sciences Mathematiques, Tome XLIX, Juin 1925 and Juillet 1925.
8 Bulletin of the American Mathematical Society, Vol. 31, Nos. 5, 6 and 7 (2 copies each).
9 Contribucion al Estudio de Las Ciencias Fisicas y Mathematicas, No. 63, Marzo 1925.
10 Journal de Mathematiques, Tome Quatrieme 1925 Fas. No. 2.
11 Journal für die reine und angewandte Mathematik [*Crelles Journal*] Band 154 heft 3.
12 Liouville's Journal, Tome IV, fascicule 1.
13 Mathematical Gazette, Vol. 12, No. 177 (3 copies).
14 Mathematische Annalen Band 94, 3/4 heft.
15 Messenger of Mathematics Vol. 54, No. 11, 12 (2 copies each).
16 Monthly Notices of the Royal Astronomical Society, Vol. 85, No. 6.
17 Nature, Vol. 115 Nos. 2899 to 2907.
18 Philosophical Magazine and Journal of Science, Vol. 50, No. 295, 296.
19 Philosophical Transactions of the Royal Society of London, Series A, Vol. 225, No. A-628.
20 Popular Astronomy, 1924, April to December ; 1925, March to July.
21 Proceedings of the Royal Society, (London) Series A, Vol. 108, No. A 746, 747.
22 Proceedings of the Edinburgh Mathematical Society, Vol. 43, Part I with Mathematical Notes, No. 23.
23 Proceedings of the London Mathematical Society, Vol. 24, Part 3.
24 Rendiconti del Circolo Matematico di Palermo XLVIII, Fasc. iii.
25 Tohoku Mathematical Journal, Vol. 11, No 4, Vol 14, Nos. 3 and 4, Vol. 25, Nos. 1 and 2 (combined).

Books.

Bombay University Calendar, 1923-24 and 1924-25, Vol. II.
Madras University Calendar, 1925-26, Vol. I.

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